Quantum Field Theory II FS24

Series 8 - Photon self-energy calculation in QED and sQED 24.04.2024

In this series, we will calculate the renormalized **photon self-energy**, to the lowest non-trivial order in perturbation theory, i.e. $\mathcal{O}(e^2)$, using **dimensional regularization** in two cases.

Case 1: QED

1. Draw the Feynman diagram for the photon self-energy (for a photon of momentum q) and prove that its expression at the lowest order in perturbation theory is

$$i\Pi^{\mu\nu}(q) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k\cdot(k+q) - m^2)}{(k^2-m^2)((k+q)^2-m^2)}. \quad (1)$$

Hint: Use the fact that $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$ *and think which terms can be dropped.*

2. Prove that this expression can be equivalently written as

$$i\Pi^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{-\frac{1}{2}g^{\mu\nu}\ell_E^2 + g^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2+\Delta)^2}, \quad (2)$$

$$\ell \equiv k + xq, \qquad \Delta = m^2 - x(1-x)q^2. \quad (3)$$

Hint: Introduce Feynman parametrization and do a Wick rotation $\ell^0 = i\ell_E^0$. *Use also the fact that, because* $g_{\mu\nu}g^{\mu\nu} = d$ *in* d*-dimensions, you could replace the numerator of a symmetric integral that contains* $\ell^\mu\ell^\nu$ *by* $\frac{1}{d}\ell^2 g^{\mu\nu}$.

3. How does this integral diverge if we cut it off at $\ell_E = \Lambda$ (leading divergent term only) ?

4. Repeat the steps in 2. in d-dimensions. Think about which terms are affected and how they change in this case.

5. Prove that

$$i\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\; i\Pi(q^2), \quad (4)$$

$$\Pi(q^2) = -\frac{8e^2}{(4\pi)^{d/2}} \int_0^1 dx\; x(1-x) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}. \quad (5)$$

Hint: Use the following results for the integrals in d *dimensions*

$$\int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-d/2}, \quad (6)$$

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-1}, \tag{7}$$

as well as the fact that $z\Gamma(z) = \Gamma(z+1)$.

6. Find the behaviour of $\Pi(q^2)$ for $d \to 4$ or equivalently $\epsilon = 4 - d \to 0$.

Hint: Leave the expression as an integral over x *and use the fact that* $\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$, *with* $\gamma \approx 0.5772$ *(Euler-Macheroni constant) and* $y^\epsilon = e^{\epsilon \log(y)} = 1 + \epsilon \log(y) + \mathcal{O}(\epsilon)$, *for* $\epsilon \to 0$.

7. Find the final expression for the renormalized photon self-energy $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$.

Case 2: Scalar QED (sQED)

8. Repeat the previous calculation, but now in the case of sQED. This time work directly in dimensional regularization.

Hint: When you have reached the final expression as an integral over the Feynman variable x*, you can further simplify it by doing an integration over parts over one term. This will bring your final expression to a form similar to the one in 5.*

9. Calculate the $\text{Im}(\hat{\Pi}(q^2 \pm i\epsilon))$ in the case $q^2 > 4m^2$. Which process (cross-section) is related to this result in each case (QED, sQED) via the unitarity theorem ?

Hint: Think about which values of x*, for fixed* $q^2 > 4m^2$*, will contribute and use the fact that* $\text{Im}(\log(-a \pm i\epsilon)) = \pm\pi$.