Quantum Field Theory II FS24

Series 9 - Anomalous magnetic moment calculation at $\mathcal{O}(\alpha)$ 08.05.2024

In this series, you will do one of the most amazing and important calculation in QED: you will calculate the anomalous magnetic moment of the muon (or electron), which is defined as $a_\mu \equiv \frac{g-2}{2} = F_2(0)$, at $\mathcal{O}(\alpha)$. This calculation gave Schwinger the 1965 Nobel prize (along Tomonaga and Feynman). So take a deep breath and let's dive into the calculation.

1. Draw the Feynman diagram for the $\mathcal{O}(e^3)$ contribution to the anomalous magnetic moment of a muon (or electron) of mass m and prove that its expression can be written as

$$F_\mu = -ie^3 \int \frac{d^4q}{(2\pi)^4} \frac{N_\mu}{D}, \tag{1}$$

$$N_\mu = \bar{u}'\gamma_\lambda(\not{p}' + \not{q} + m)\gamma_\mu(\not{p} + \not{q} + m)\gamma^\lambda u, \tag{2}$$

$$D = [(p' + q)^2 - m^2][(p + q)^2 - m^2][q^2]. \tag{3}$$

Careful that there is no $(-i)$ at the external field vertex, because it was factored out in the definition of F_μ.

2. Prove that N_μ can be simplified to the following expression

$$N_\mu = 4p' \cdot p(\bar{u}'\gamma_\mu u) + 2(\bar{u}'\gamma_\mu \not{q}\not{p}' u) + 2(\bar{u}'\not{p}\not{q}\gamma_\mu u) + (\bar{u}'\gamma_\lambda \not{q}\gamma_\mu \not{q}\gamma^\lambda u). \tag{4}$$

and argue why the first term in this expression can be dropped for this specific calculation.

Hint: Use the Dirac equation for u and $\bar{u}'$, as well as the anti-commutation relations for the gamma matrices.

3. Prove that

$$\frac{1}{D} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[q'^2 - m^2(x+y)^2]^3} + \mathcal{O}(k^2), \tag{5}$$

with $q' \equiv q + xp' + yp$.

Hint: Use Feynman parametrization as well as on-shell conditions for the muon (or electron). After simplifying the whole expression, use the fact that we are interested in the region $k^2 \to 0$.

4. Use the change of variable from 3. and prove that N_μ can be written in the following form

$$N_\mu = -2\bar{u}'(y\gamma_\mu \not{p}\not{p}' + x\not{p}\not{p}'\gamma_\mu)u + \bar{u}'\gamma_\lambda(x\not{p}' + y\not{p})\gamma_\mu(x\not{p}' + y\not{p})\gamma^\lambda u + \bar{u}'\gamma_\lambda \not{q}'\gamma_\mu \not{q}'\gamma^\lambda u. \tag{6}$$

Hint: Drop terms linear in q', use on-shell conditions for the muon (or electron), and remember also that terms proportional to $\bar{u}'\gamma_\mu u$ can be dropped, as they contribute only to F_1.

5. Bring (6) to the following form

$$N_\mu = 2\bar{u}'[my\gamma_\mu \not{p}' + mx\not{p}\gamma_\mu - (x\not{p}' + y\not{p})\gamma_\mu(x\not{p}' + y\not{p})]u. \tag{7}$$

Hint: Use the Dirac equation, the identities $\gamma_\lambda\gamma_\mu\gamma^\lambda = -2\gamma_\mu$, $\gamma_\lambda \not{a}\not{b}\not{c}\gamma^\lambda = -2\not{c}\not{b}\not{a}$, and think how you can treat the q'^2 term from the hints of series 7. Terms proportional to $\bar{u}'\gamma_\mu u$ and of $\mathcal{O}(k^2)$ can be dropped.

6. Make the replacement $x \to \frac{1}{2}(x+y)$ and $y \to \frac{1}{2}(x+y)$, since we are integrating over a a symmetric region under the exchange $x \leftrightarrow y$ (antisymmetric parts under this exchange will drop) and prove that (7) can be simplified further to the following expression

$$N_\mu = -2im(\bar{u}'\sigma_{\mu\nu}u)k^\nu[(x+y) - (x+y)^2]. \tag{8}$$

Hint: Use momentum conservation to turn the p' on the right over to p as well as the p on the left to p'. Use also the hints from the previous questions, as well as the definition $[\gamma_\mu, \gamma_\nu] = -2i\sigma_{\mu\nu}$.

7. Put everything together and prove that $F_2(k^2)|_{k^2\to 0} \equiv F_2(0)$ is

$$F_2(0) = 8ie^2m^2 \int_0^1 dx \int_0^{1-x} dy[(x+y) - (x+y)^2] \int \frac{d^4q'}{(2\pi)^4} \frac{1}{[q'^2 - m^2(x+y)^2]^3}. \tag{9}$$

Hint: Remember the decomposition of F_μ in F_1 and F_2.

8. Prove that $F_2(0) = \dfrac{e^2}{8\pi^2}$ and thus that the anomalous magnetic moment of the muon (or electron) at $\mathcal{O}(\alpha)$ is $a_\mu = \dfrac{\alpha}{2\pi}$.

Hint: Remember that

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - a^2]^3} = -\frac{i}{32\pi^2 a^2}. \tag{10}$$

Congratulations! You have just completed a calculation worthy of a Nobel prize!