Quantum Field Theory II FS24

Series 10 - One-loop muon/electron vertex function at $\mathcal{O}(\alpha)$. 15.05.2024

In this series you will calculate analytically the muon/electron vertex function at $\mathcal{O}(\alpha)$, but this time, you will focus more on the $\mathcal{O}(\alpha)$ correction to the form factor $F_1(q^2)$. A large part of the procedure from series 9 ($\mathcal{O}(a)$ correction to form factor $F_2(q^2)$) will be repeated here, but it is still an important exercise to do once more, so you master one-loop calculations to the maximum.

1. Draw the Feynman diagram for the $\mathcal{O}(e^2)$ correction to the muon/electron vertex function and prove that its expression can be written as

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')[\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu]u(p)}{[(k-p)^2 + i\epsilon][k'^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]}, \tag{1}$$

with $k' = k + q$.

Hint: Use the identity $\gamma_\lambda \gamma_\mu \gamma^\lambda = -2\gamma_\mu$ and think which of terms drop.

2. Using Feynman parametrization, prove that

$$\frac{1}{[(k-p)^2 + i\epsilon][k'^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{(\ell^2 - \Delta + i\epsilon)^3}, \tag{2}$$

with $\ell \equiv k + yq - zp$ and $\Delta \equiv -xyq^2 + (1-z)^2 m^2$.

3. Use the change of variables from 2. and prove that the numerator can be simplified to

$$\begin{aligned}\text{Numerator} = \bar{u}(p')\Big[&\gamma^\mu\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right) \\ &+ (p^\mu + p'^\mu)mz(z-1) + q^\mu m(z-2)(x-y)\Big]u(p).\end{aligned} \tag{3}$$

What happens to the term proportional to q^μ and why ?

Hint: Group the numerator in terms of $A\gamma^\mu + B(p^\mu + p'^\mu) + Cq^\mu$ and think how to treat terms proportional to $\ell^\mu \ell^\nu$. You will also need the anticommutation relations for gamma matrices, as well as the Dirac equation. Also remember that $x + y + z = 1$.

4. Prove that

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2 &\int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{(\ell^2 - \Delta + i\epsilon)^3} \\ &\times \bar{u}(p')\Big[\gamma^\mu\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right) \\ &+ i\frac{\sigma^{\mu\nu}q_\nu}{2m}2m^2 z(z-1)\Big]u(p).\end{aligned} \tag{4}$$

Hint: Use the Gordon identity.

You can now clearly see the decomposition in a term proportional to γ^μ and one proportional to $\sigma^{\mu\nu}$, which are related to the form factors F_1 and F_2 respectively.

For the last step you have to perform the $d^4\ell$ integrals. You have already done similar calculations in the previous series in **dimensional regularization**. You could of course proceed like this in this case too, if you want, however this time it would be beneficial to do the same calculation using **Pauli-Villars regularization**, so that you see this approach too in practice.

The problematic integral is the following one

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}}, \tag{5}$$

because for $m = 3$, which is our case, it becomes divergent.

One way to tackle it is to do the following replacement in the photon propagator in the initial expression (1)

$$\frac{1}{(k-p)^2 + i\epsilon} \to \frac{1}{(k-p)^2 + i\epsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\epsilon}, \tag{6}$$

with Λ a large mass. In this way the integrand is unaffected for small k (since Λ is large), but cuts off smoothly when $k \geq \Lambda$.

5. Repeat the steps in 1. and 2. and prove that the only thing that changes in the integral with the heavy photon is the following

$$\Delta \to \Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2. \tag{7}$$

6. Prove that the final expression for the one-loop vertex correction becomes

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = {} & \frac{\alpha}{2\pi} \int_0^1 dxdydz\delta(x+y+z-1) \\ & \times \bar{u}(p') \Bigg(\gamma^\mu \Bigg[\log \frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta} \Big((1-x)(1-y)q^2 + (1-4z+z^2)m^2 \Big) \Bigg] \\ & + i \frac{\sigma^{\mu\nu} q_\nu}{2m} \frac{1}{\Delta} 2m^2 z(z-1) \Bigg) u(p).\end{aligned} \tag{8}$$

Hint: Use the following expressions for the integrals

$$\int \frac{d^4\ell}{(2\pi)^4} \left(\frac{\ell^2}{[\ell^2 - \Delta]^3} - \frac{\ell^2}{[\ell^2 - \Delta_\Lambda]^3} \right) = \frac{i}{(4\pi)^2} \log \frac{\Delta_\Lambda}{\Delta}, \tag{9}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^3} = \frac{1}{2i(4\pi)^2} \frac{1}{\Delta}. \tag{10}$$

Observe also that the convergent terms - which are proportional to (10) *- are modified by terms of order* Λ^{-2}*, which you can ignore.*

Let's now focus on the term proportional to γ^μ, which has to do with the form factor $F_1(q^2)$. In the expression (8), you can see that there is a UV divergence, due to the logarithm, and an also an IR divergence, due to the term proportional to Δ^{-1}. Both have to be addressed.

- The UV divergence can be taken care of by making the substitution $\delta F_1(q^2) \to \delta F_1(q^2) - \delta F_1(0)$.

- The IR divergence can be cured by pretending that the photon has a small non-zero mass μ, so the denominator related to the photon $(k-p)^2$ becomes $(k-p)^2 - \mu^2$, which makes Δ become $\Delta + z\mu^2$ (algebraically the same computation as in 5.).

7. Using this information, prove that the form factor $F_1(q^2)$ takes the following form at $\mathcal{O}(\alpha)$

$$\begin{aligned} F_1(q^2) = 1 &+ \frac{\alpha}{2\pi} \int_0^1 dxdydz\delta(x+y+z-1) \\ &\times \left[\log \frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy} + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} \right. \\ &\left. - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z} \right] + \mathcal{O}(\alpha^2). \end{aligned} \tag{11}$$

8. (Optional) Find the expression for the form factor $F_2(q^2)$ at $\mathcal{O}(\alpha)$, calculate the anomalous magnetic moment of the muon/electron and verify the result you found in the same calculation in series 9.

Hint: Focus on the term proportional to $\sigma^{\mu\nu}$ *in* (8).