QED with a massive photon

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu + \bar{\psi}(i\not{\partial} - m - e\not{A})\psi$$

Photon propagator:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{i}{k^2-\mu^2+i\varepsilon}\left[-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2}\right]$$

Massless QED Lagrangian with gauge-fixing term.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m - e\not{A})\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Photon propagator:

$$\tilde{D}_\xi^{\mu\nu}(k) = \frac{i}{k^2+i\varepsilon}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) - \frac{i\xi}{k^2+i\varepsilon}\left(\frac{k^\mu k^\nu}{k^2}\right)$$

($\xi = 1 \Rightarrow$ Feynman gauge; $\xi = 0 \Rightarrow$ Landau gauge; $\xi = 3 \Rightarrow$ Yennie-Fried gauge)

This we have obtained with the Faddeev-Popov trick and we have seen how to derive Feynman rules (in the path-integral approach) and make calculations in perturbation theory. We now want to address the question of renormalisation. Proceeding as we did before, we would introduce renormalized fields of the form:

$$A'_\mu = Z_3^{-1/2} A_\mu \quad ; \quad \psi' = Z_2^{-1/2}\psi$$

$$\mathcal{L} = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A'^\mu)^2 + \bar{\psi}'(i\not{\partial} - m - ie\not{A}')\psi + \mathcal{L}_{CT}$$

What should we include in $\mathcal{L}_{CT}$? (I'll drop the prime from now on)

$$\mathcal{L}_{CT} = A \overbrace{A_\mu A^\mu}^{d=2} + B \overbrace{\bar{\psi}\psi}^{d=3} + C \bar{\psi} \overset{d=4}{\not{\partial}} \psi + D \bar{\psi} e \not{A} \psi$$
$$+ E F_{\mu\nu} F^{\mu\nu} + F (\partial_\mu A^\mu)^2 + G (A_\mu A^\mu)^2$$

A and G are not gauge invariant $(A_\mu \to A_\mu + \partial_\mu \chi)$

if we need them, we have to rethink the whole theory.

Gauge invariance is also ensured by the combination

$$\bar{\psi} i (\not{\partial} - e\not{A}) \psi$$

which would indicate that we need $C = -D$. However, since we have left the constant e explicitly, a violation of this relation could be compensated by a redefinition of e. If we had more than one field and more than one e, it is only a relation of the form $C = -D$ which could explain why the different couplings would renormalize in the same way.

A renormalization of ξ would not make much sense either: no observable can depend on ξ, why would we be forced to redefine ξ? $\Rightarrow F = 0$.

So, ideally we would like to have:

$$A = F = G = 0 \quad , \quad C = -D$$

satisfied in our renormalization procedure. Is this the case?

The answer is yes, but in order to prove it we will have to introduce some new concepts and work at a somewhat formal level. Before doing this, I want to show in some concrete calculations that these relations are satisfied. In particular if we focus on C=-D we see that we must have a relation between the two objects:

Coleman argues as follows:

consider $j_\mu = \bar\psi \gamma_\mu \psi$ and its equal-time commutator:

$$\left[j_0(\vec{x},t), \psi(\vec{y},t) \right] = -\delta^3(\vec{x}-\vec{y})\,\psi(\vec{y},t)$$

Since we have $\partial_\mu j^\mu = 0$, the divergence of the time-ordered product:

$$\partial_x^\mu \langle 0 | T\, j_\mu(x) \psi(y) \bar\psi(z) | 0 \rangle$$

gets contributions only from the time derivative acting on θ-functions. The final result is:

$$\partial_x^\mu \langle 0 | T\, j_\mu(x) \psi(y) \bar\psi(z) | 0 \rangle = -\delta^4(x-y) \langle 0 | T \psi(y) \bar\psi(z) | 0 \rangle + \delta^4(x-z) \langle 0 | T \psi(y) \bar\psi(z) | 0 \rangle$$

which, in some way, seems to express the desired relation.

This kind of relations, which follow from current conservation, are called Ward identities and are essential in ensuring that gauge invariance survives renormalization.

We will now discuss one example of such a Ward identity and prove that it is satisfied in perturbative calculations to any order. We will do so by looking at it from the point of view of Feynman diagrams.

If we consider a Green's function involving $j^\mu(x)$

$$G_\mu^{(n+1)}(x_1, \dots x, \dots, x_n) = \langle 0 | T \cdots j_\mu(x) \cdots | 0 \rangle$$

and then its Fourier transform

$$\tilde{G}_\mu^{(n+1)}(k_1, \dots, k, \dots, k_n)$$

$$G_\mu^{(n+1)}(x_1, \dots, x_n) = \int \frac{d^4k_1 \dots d^4k_n\, d^4k}{(2\pi)^{4(n+1)}} e^{-ikx} e^{-i(k_1x_1 \dots k_nx_n)} \tilde{G}_\mu^{(n+1)}(k_1, \dots, k_n)$$

we see that current conservation $\partial_\mu j^\mu(x) = 0$ has consequences for the product: $k^\mu \tilde{G}_\mu^{(n+1)}(k_1, \dots, k_n)$

which, as in the example above, will be related to Green's functions without the current. A particularly simple version thereof is obtained by considering a matrix element of the current between an initial and final state:

$$M_{fi}^\mu(k) = \int d^4x\, e^{ikx} \langle f | j^\mu(x) | i \rangle$$

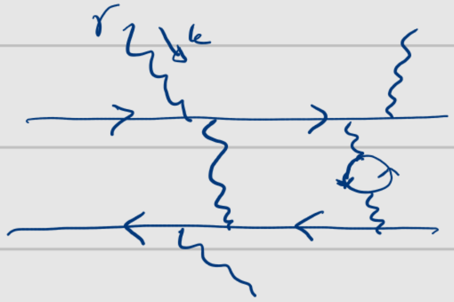
which gives the matrix element of the process $i\gamma \to f$

$$M(i\gamma \to f) = \varepsilon_\mu(k) M_{fi}^\mu(k)$$

In this case the consequence of current conservation is very simple:

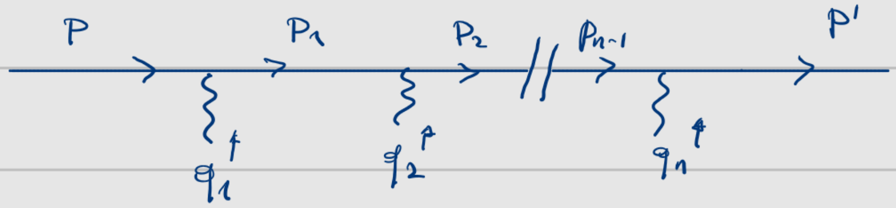
$$k_\mu M_{fi}^\mu(k) = 0$$

At tree level it is very easy to check that this holds. More complicated is the question: Does this relation hold even if one considers loop diagrams? We will answer the question by looking at a generic diagram in QED:

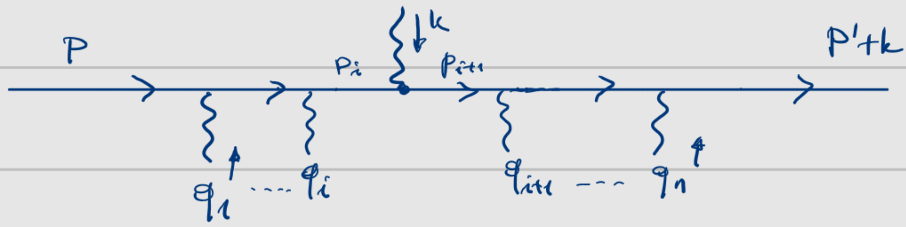


By removing the photon $\gamma(k)$ we obtain a simpler amplitude $\mathcal{M}_0$. From the same amplitude we can obtain more contributions to $\mathcal{M}^\mu(k)$ by hooking the photon to any other possible position on the same line and any other possible position on any other line. We will now consider what is the effect of hooking the photon to any line by first considering external and then internal lines.

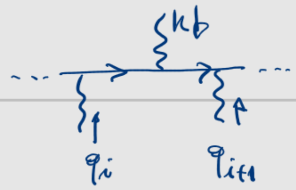
External line.



insertion of $\gamma(k)$



Vertex $\quad -ie\varepsilon_\mu\gamma^\mu \longrightarrow -iek^\mu\gamma_\mu = -ie\left[(\not p_i+\not k-m)-(\not p_i-m)\right]$

$\Longrightarrow$ 

$$= \cdots\left(\frac{i}{\not p_{i+1}+\not k-m}\right)\gamma^{\lambda_{i+1}}\left(\frac{i}{\not p_i+\not k-m}\right)(-ie)\left[(\not p_i+\not k-m)-(\not p_i-m)\right]\times$$

$$\times\left(\frac{i}{\not p_i-m}\right)\gamma^{\lambda_i}\left(\frac{i}{\not p_{i-1}-m}\right)\cdots$$

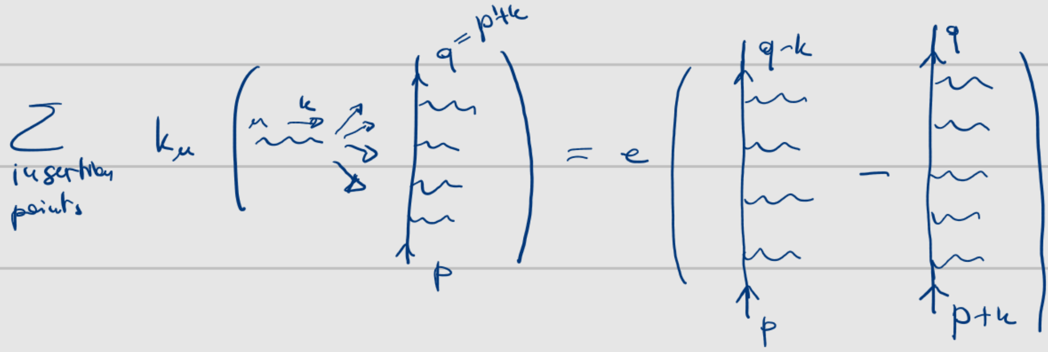
$$= \cdots \left(\frac{i}{\not{p}_{i+1} + \not{k} - m} \right) \gamma^{\lambda_{i+1}} (-ie) \left[\frac{i}{\not{p}_i - m} - \frac{i}{\not{p}_i + \not{k} - m} \right] \gamma^{\lambda_i} \frac{i}{\not{p}_{i-1} - m} \cdots$$

If we move the photon $\gamma(k)$ to the left by one position we get:

$$\cdots \left(\frac{i}{\not{p}_{i+1} + \not{k} - m} \right) \gamma^{\lambda_{i+1}} \left(\frac{i}{\not{p}_i + \not{k} - m} \right) \gamma^{\lambda_i} (-ie) \left[\frac{i}{\not{p}_{i-1} - m} - \frac{i}{\not{p}_{i-1} + \not{k} - m} \right] \gamma^{\lambda_{i-1}} \cdots$$

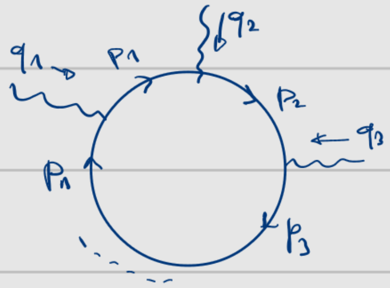
and we see that the first term in square brackets in the latter expression cancels with the second in the former expression.

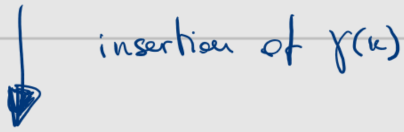
In the sum over all possible insertions of $\gamma(k)$ it is only one contribution from each of the external insertions which survives:

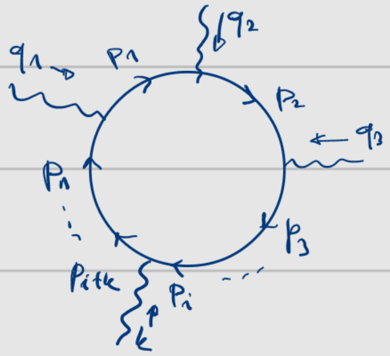


Note that the true momenta of the external fermions are p and q, even if the two terms in the outcome have as external propagators two with momenta p and $q-k$ and $p+k$ and q, respectively. But in order to get the amplitude by applying LSZ, I have to take the residue of $\frac{1}{\not{p}-m} \cdot \frac{1}{\not{q}-m}$, which none of the two terms has. So, the contribution of any line ending as external to the amplitude $k^\mu M_\mu^{ext}(k) = 0$.

Internal fermion line.







We can now apply the same argument as above. For example, for the diagram where the insertion occurs between q_1 and q_2 we get

$$-e\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{i}{\not p_n+\not k-m}\right)\gamma^{\lambda_n}\cdots\left(\frac{i}{\not p_2+\not k-m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1-m}-\frac{i}{\not p_1+\not k-m}\right)\gamma^{\lambda_1}\right]$$

whereas if we insert it between q_2 and q_3:

$$-e\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{i}{\not p_n+\not k-m}\right)\gamma^{\lambda_n}\cdots\gamma^{\lambda_3}\left(\frac{i}{\not p_2-m}-\frac{i}{\not p_2+\not k-m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1-m}\right)\gamma^{\lambda_1}\right]$$

so that the second term here cancels the first above.

Continuing the procedure until all possible insertions have been carried out, we end up with:

$$-e\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{i}{\not p_n - m}\right)\gamma^{\lambda_n}\cdots\gamma^{\lambda_2}\left(\frac{i}{\not p_1 - m}\right)\gamma^{\lambda_1} - \underbrace{\left(\frac{i}{\not p_n + \not k - m}\right)\gamma^{\lambda_n}\cdots\gamma^{\lambda_2}\left(\frac{i}{\not p_1 + \not k - m}\right)\gamma^{\lambda_1}}_{\text{shift of the integr. variable: } p_1 + k \to p'}\right]$$

$$= 0$$

$$\Longrightarrow \qquad k^\mu M_\mu^{int}(k) = 0$$

Which concludes our proof.