After the generating functional of all Green's functions as well as that of the connected Green's functions:

$$Z[J] = N\int \mathcal{D}\phi \, e^{iS[\phi,J]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n J(x_1)\dots J(x_n)\, G^{(n)}(x_1,\dots,x_n)$$

$$\equiv e^{iW[J]}$$

for which we have:

$$iW[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n J(x_1)\dots J(x_n)\, G_c^{(n)}(x_1,\dots,x_n)$$

we are now going to introduce the GF of all 1PI Green's functions.

Before we start with its definition and the relation to the GF $W[J]$ it is useful to make some observation about the loop expansion in $W[J]$ and power-counting in $\hbar$. For this we need to reintroduce $\hbar$:

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi,J]/\hbar}$$

If we consider a Feynman diagram we can count powers of $\hbar$ as follows:

- propagators are the inverse of the quadratic part of the action $\Rightarrow \tilde{D} \sim \hbar$
- vertices are proportional to $\frac{1}{\hbar}$

$$\Rightarrow \quad G \sim \hbar^{(I-V)}$$

G = Feynman diagram or Graph

For a <u>connected</u> graph the number of loops is equal to the number of internal lines minus the (number of vertices − 1) $\Rightarrow L = I - V + 1$

↑ d^4k_i

δ-functions at each vertex

↑ δ-function which must remain for total mom. conserv.

$$G_c \sim \hbar^{(L-1)}$$

$\Rightarrow$ For $W_\hbar[J] = -i \ln Z_\hbar$ the expansion in $\hbar$ exactly corresponds to the expansion in the number of loops:

$$W_\hbar[J] = \frac{1}{\hbar}\left[W_0[J] + \hbar W_1[J] + \hbar^2 W_2[J] + \dots\right]$$

* * *

Effective action

Our goal is to define a GF analogously to the two we defined above, which now has as coefficient of its variable the 1PI "diagrams". In this case it wouldn't make sense to have J, the external source as the variable of the GF, and it is more natural to have the field as such variable. As we will see, it will be the classical field: $\bar{\Phi}$, so that

$$i\Gamma[\bar{\Phi}] = i\sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1 \dots d^4x_n\, \Gamma^{(n)}(x_1,\dots,x_n)\bar{\Phi}(x_1)\dots\bar{\Phi}(x_n)$$

$$\Gamma^{(n)}(x_1,\dots,x_n) = \int\frac{d^4p_1}{(2\pi)^4}\dots\frac{d^4p_n}{(2\pi)^4}e^{i(p_1x_1+\dots+p_nx_n)}\underbrace{\tilde{\Gamma}(p_1,\dots,p_n)}_{\substack{\uparrow\\ \text{1PI diagrams.}}}(2\pi)^4\delta^4(p_1+\dots+p_n)$$

For $n>2$ $i\tilde{\Gamma}(p_1,\dots,p_n)$ correspond to the 1PI diagrams we discussed, as they don't include any δ-function and all external legs have been amputated.

For $n=2$ there is a caveat:

$$\tilde{G}^{(2)}(p,p') = (2\pi)^4\delta^4(p+p')\tilde{D}(p)$$

amputating this Green's function means to multiply it twice with the inverse propagator and removing the δ-function

$$\tilde{\Gamma}^{(2)}(p,-p) = \tilde{D}(p)^{-1}\,\tilde{D}(p)\,\tilde{D}^{-1}(p) = \tilde{D}^{-1}(p)$$

This is not quite the 1PI $\tilde{\Pi}(p^2)$ we have defined  and which gets resummed into:

$$\tilde{D}(p) = \frac{i}{p^2-\mu^2-\tilde{\Pi}(p^2)+i\varepsilon}$$

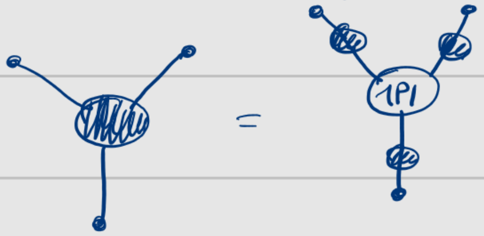
$$i\tilde{\Gamma}^{(2)}(p,-p) = i(p^2-\mu^2) - i\tilde{\Pi}(p^2)$$

as it includes the "tree-level" contribution to the inverse propagator and not just the 1PI loop contributions. In this way we have a consistent algebraic definition rather than a graphical one.

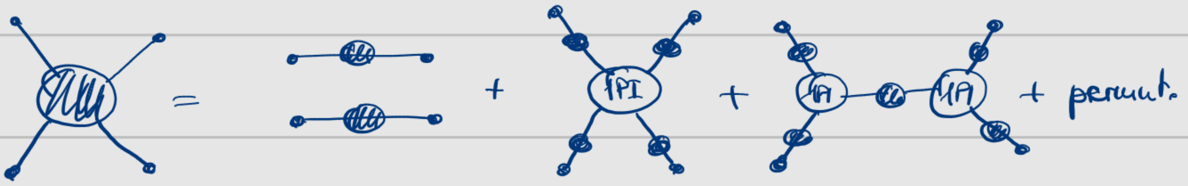
What is the connection between $\Gamma[\bar{\Phi}]$ and $W[J]$?

We can understand this if we work with diagrams:

3-point function



4-point function



⟹ the example of the 4-point function shows that if I include the vertices of the effective action in tree-diagrams only, what I get corresponds to the Green's functions. But, as we have seen above, the tree-level approximation is the leading order in an expansion in $\hbar$ in the path integral. So we can formulate the relation between Γ and W as follows:

$$Z[J] = \exp\left\{\frac{i}{\hbar}\Big(W[J] + O(\hbar)\Big)\right\} = N\int \mathcal{D}\bar{\Phi}\, \exp\left\{\frac{i}{\hbar}\left[\Gamma[\bar{\Phi}] + \int d^4x\, J\bar{\Phi}\right]\right\}$$

This shows that $S[\bar{\Phi}] = \Gamma[\bar{\Phi}]\Big|_{\text{tree}}$

Now that we have established a relation between W and Γ, even if somewhat formal, we should make it more concrete and explicit. How can we get only the first term in this expansion in $\hbar$?
We can resort to the method of the stationary phase: as we send $\hbar \to 0$ the argument of the exponential becomes very large and tiny variations get amplified leading to wild oscillations which vanish when integrated over $\Rightarrow$ the only contribution comes from the point where the phase is stationary w.r.t. the integration variable. The condition is expressed as:

$$\frac{\delta\Gamma[\bar{\Phi}]}{\delta\bar{\Phi}(x)} = -J(x)$$

Since $J(x)$ is an arbitrary function this equation has to be read as determining $\bar{\Phi}_J(x)$. If we plug in this solution into $\Gamma[\phi]$ we get $W[J]$:

$$W[J] = \Gamma[\bar{\Phi}_J] + \int d^4x\, J(x)\bar{\Phi}_J(x)$$

which is, however, the opposite of what we want: we would like to calculate Γ if we know W. But the relation between the two functionals is that of a Legendre transformation, which we just need to invert:

if we calculate the variation of $W[J]$ with respect to J we obtain:

$$\delta W = \int d^4y \left[\frac{\delta\Gamma}{\delta\bar{\Phi}(y)}\, \delta\bar{\Phi}(y) + \delta J(y)\,\bar{\Phi}(y) + J(y)\,\delta\bar{\Phi}(y) \right]$$

$$= \int d^4y \left[-\cancel{J(y)\delta\bar{\Phi}(y)} + \delta J(y)\bar{\Phi}(y) + \cancel{J(y)\delta\bar{\Phi}(y)} \right]$$

$$= \int d^4y\, \delta J(y)\bar{\Phi}(y)$$

$$\Rightarrow \quad \frac{\delta W}{\delta J(x)} = \bar{\Phi}(x)$$

which now determines $J_\phi(x)$ as a function of $\bar{\Phi}$.

$$\frac{\delta\Gamma}{\delta\phi} = -J \quad \Rightarrow \quad W[J] = \Gamma[\bar{\Phi}_J] + \int d^4x\, J(x)\bar{\Phi}_J(x)$$

$$\frac{\delta W}{\delta J} = \bar{\Phi}(x) \quad \Rightarrow \quad \Gamma[\bar{\Phi}] = W[J_{\bar{\Phi}}] - \int d^4x\, J_{\bar{\Phi}}(x)\bar{\Phi}(x)$$

Let us now explicitly differentiate Z wrt J:

$$\frac{\delta Z[J]}{\delta J(x)} = N \int \mathcal{D}\phi \, e^{i\left[S + \int d^4y \, J(y)\phi(y)\right]} \phi(x)$$

$$\frac{\delta Z[J]}{\delta J(x)} = e^{iW[J]} \, i \frac{\delta W}{\delta J(x)} = e^{iW[J]} \, i \bar{\Phi}(x)$$

$$\Rightarrow \; iN \int \mathcal{D}\phi \, \phi(x) \, e^{i[S+J\phi]} = \bar{\Phi}(x) \; iN \int \mathcal{D}\phi \, e^{i[S+J\phi]}$$

$$\Rightarrow \; \bar{\Phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\int \mathcal{D}\phi \, \phi(x) \, e^{i[S+J\phi]}}{\int \mathcal{D}\phi \, e^{i[S+J\phi]}} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}$$

Note that if $f(\phi)$ is a linear function of ϕ: $f(\phi) = \alpha\phi + \beta$

then

$$N \int \mathcal{D}\phi \, f(\phi) \, e^{i[S+J\phi]} = f(\bar{\Phi}) \; N \int \mathcal{D}\phi \, e^{i[S+J\phi]} .$$

Effective action and QED.

Consider a gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \delta\chi$$

$$\psi \rightarrow \psi - ie\psi\delta\chi$$

$$\bar{\psi} \rightarrow \bar{\psi} + ie\bar{\psi}\delta\chi$$

Define

$$\underline{\Phi} = \begin{pmatrix} \psi \\ \bar{\psi} \\ A_\mu \end{pmatrix}$$ and combine the gauge transf. into a single one:

$$\Phi \longrightarrow \Phi' = \Phi + A(\Phi)\,\delta\chi$$

Note that $A(\Phi)$ is at most first order in Φ.

$$S[\Phi] = S_{GI}[\Phi] + S_{GF}[\Phi] = S_{GI}[\Phi] - \frac{1}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2$$

After a gauge transformation we get

$$S[\Phi'] = S[\Phi] + \delta S[\Phi] = S[\Phi] - \frac{1}{\xi}\int d^4x\,(\partial_\mu A^\mu)\Box\,\delta\chi$$

$$\equiv S[\Phi] + \int d^4x\, B(\Phi)\,\delta\chi$$

where $B(\Phi) = -\frac{1}{\xi}\partial_\mu A^\mu \Box$

Since the shift in Φ leaves the measure invariant (A_μ gets shifted while ψ and $\bar\psi$ get rotated $\rightarrow \det T = 1$) we can argue as follows:

$$e^{iW[J]} = N\int \mathcal{D}\Phi\; e^{i[S[\Phi] + J\Phi]} \rightarrow N\int \mathcal{D}\Phi'\, e^{i[S[\Phi'] + J\Phi']}$$

$$= N\int \mathcal{D}\Phi\; e^{i\left[S[\Phi] + B(\Phi)\delta\chi + J(\Phi + A(\Phi)\delta\chi)\right]}$$

$$= e^{iW[J]} + N\int \mathcal{D}\Phi\; e^{i[S[\Phi]+J\Phi]}\left\{ i\int d^4y\,\big(B(\Phi) + JA(\Phi)\big)\delta\chi \right\}$$

$$= e^{iW[J]}$$

because the integral cannot change for a change of variable.

$$\Rightarrow \quad N\int \mathcal{D}\Phi\; e^{i[S[\Phi]+J\Phi]}\left\{ i\int d^4y\,\big(B(\Phi) + JA(\Phi)\big)\delta\chi \right\} = 0$$

But as we have seen above, for a linear function, the integration just fixes the integration variable at its mean value:

$$N\int \mathcal{D}\Phi\; e^{i[S[\Phi]+J\Phi]}\left\{ i\int d^4y\,\big(B(\Phi) + JA(\Phi)\big)\delta\chi \right\} = e^{iW[J]}\left\{ i\int d^4y\,\big(B(\bar\Phi) + JA(\bar\Phi)\big)\delta\chi \right\}$$

we can drop $e^{iW[J]}$ and replace $J(x)$ with $-\frac{\delta\Gamma}{\delta\bar{\Phi}(x)}$, which gives:

$$\int d^4y \, \frac{\delta\Gamma}{\delta\bar{\Phi}} A(\bar{\Phi})\delta\chi = \int d^4y \, B(\bar{\Phi})\delta\chi(y)$$

But the LHS is nothing but the variation of Γ under a gauge transformation

$$\delta\Gamma[\bar{\Phi}] = \int d^4x \, \frac{\delta\Gamma[\bar{\Phi}]}{\delta\bar{\Phi}}\delta\bar{\Phi} = \int d^4x \, \frac{\delta\Gamma[\bar{\Phi}]}{\delta\bar{\Phi}} A(\bar{\Phi})\delta\chi$$

from which we conclude that

$$\Gamma[\bar{\Phi}] \to \Gamma[\bar{\Phi}] + \delta\Gamma[\bar{\Phi}] = \Gamma[\bar{\Phi}] + \underbrace{\int d^4y \, B(\bar{\Phi})\delta\chi(y)}_{\delta S_{GF}[\bar{\Phi}]}$$

This is a remarkable and very important result which is worthwhile writing in a framed box:

$$\boxed{\delta\Gamma[\bar{\Phi}] = \delta S_{GF}[\bar{\Phi}]}$$

After a gauge transformation the effective action changes exactly as the gauge-fixing, local, part of the classical action changes. Otherwise is the effective action gauge invariant:

$$\Gamma[\bar{\Phi}] = \Gamma_{GI}[\bar{\Phi}] + S_{GF}[\bar{\Phi}]$$

Let us now look at the details of what happens in QED.

$$S_{QED} = S_{GI} + S_{GF} = \int d^4x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right\}$$

A gauge transformation is given by:

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \delta\chi \\ \psi &\to \psi - ie\psi\,\delta\chi \\ \bar\psi &\to \bar\psi + ie\bar\psi\,\delta\chi \end{aligned}$$

The change in the effective action is given by

$$\delta\Gamma = \int d^4x\, \delta\chi(x) \left\{ -ie \frac{\delta\Gamma}{\delta\psi(x)}\psi(x) + ie\bar\psi(x)\frac{\delta\Gamma}{\delta\bar\psi(x)} - \partial^\mu \frac{\delta\Gamma}{\delta A_\mu(x)} \right\}.$$

whereas, as we calculated above already:

$$\delta S_{GF} = -\frac{1}{\xi}\int d^4x\, \partial_\mu A^\mu \,\Box\, \delta\chi(x) = -\frac{1}{\xi}\int d^4x\, \delta\chi(x)\, \Box\, \partial_\mu A^\mu(x)$$

Setting the two expressions equal and observing that the identity must hold for any $\delta\chi(x)$ ($\Rightarrow$ it must be valid for its coefficient in the integrand), we conclude that:

$$ie\left(\bar\psi\frac{\delta\Gamma}{\delta\bar\psi} - \frac{\delta\Gamma}{\delta\psi}\psi\right) - \partial^\mu\frac{\delta\Gamma}{\delta A_\mu} = -\frac{1}{\xi}\Box\,\partial_\mu A^\mu$$

If we expand Γ in the fields:

$$\Gamma = \sum_n \int d^4x_1 \ldots d^4x_n\, \Gamma^{(n)}(x_1, \ldots, x_n)\, \phi(x_1) \cdots \phi(x_n)$$

and for our specific case: $\Gamma^{(n_\psi, n_{\bar\psi}, n_{A_\mu})}$

$$\begin{aligned}\Gamma = \int d^4x\, \Gamma^{(0,0,1)}_\mu(x) A^\mu(x) &+ \int d^4x_1 d^4x_2\, \bar\psi(x_1)\Gamma^{(1,1,0)}(x_1,x_2)\psi(x_2) \\ &+ \int d^4x_1 d^4x_2\, A^\mu(x_1) A^\nu(x_2)\, \Gamma^{(0,0,2)}_{\mu\nu}(x_1,x_2) \\ &+ \int d^4x_1 d^4x_2 d^4x_3\, \bar\psi(x_1)\Gamma^{(1,1,1)}_\mu(x_1,x_2,x_3)\psi(x_2)A^\mu(x_3) + \ldots\end{aligned}$$

we get a relation involving all possible effective vertex functions (the $\Gamma^{(n)}$).
However, we can split this into infinitely many relations by taking more derivatives wrt. the fields and then setting the fields to zero.
Note that if I take derivatives wrt. ψ or $\bar\psi$ the RHS drops out.

Implications for the renormalization program in QED.

In a few words, the implications of the just derived Ward identities for the renormalization program of QED are: in order to renormalize QED only gauge invariant counterterms are required, provided the gauge-fixing term is no more than quadratic in the fields.

The proof of this statement goes by induction and relies on the BPHZ program, which I am now going to outline briefly.

The BPHZ program.

a. Define D the superficial degree of divergence of a diagram:

$$D = P_N - P_D$$ (powers of integration variable in the numerator minus powers in the denominator).

b. Carry out a Taylor expansion of the Feynman diagram around momenta = 0.

Then proceed as follows:

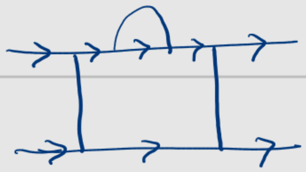
1. start your calculations in perturbation theory until you find a 1PI diagram with $D \geq 0$.

2. add counterterms to the Lagrangian to cancel the divergent terms in the Taylor expansion of the diagram which are of order $\leq D$.

3. Go back to step 1. with the new Lagrangian $\mathcal{L}' = \mathcal{L} + \mathcal{L}_{CT}$

The algorithm was proposed by Bogoliubov and Parasiuk (1957). Hepp showed (1966) that the algorithm removes all divergences (for a massive theory) and that all Green's functions remain finite after the cut-off is removed to all orders in perturbation theory. Zimmerman later showed (1969) that all ultraviolet divergences are removed by the algorithm.

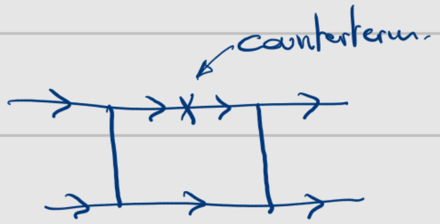
An important observation about this program is the following. Divergences may hide in diagrams which have $D<0$. For example:



$$D = 4\times\overset{\text{loops}}{2} - \underset{\substack{\uparrow\\ \text{scalar}\\ \text{propagators}}}{3\times 2} - \underset{\substack{\uparrow\\ \text{fermion}\\ \text{propag.}}}{4\times 1} = 8-10=-2$$

but the self-energy of the fermion propagator in the higher part of the diagram is divergent.

However, this divergence would have been removed in an earlier step of the BPHZ program, and at this level would be removed by the diagram



so that the sum of the two diagrams indeed remains finite.

* * *

Back to QED. Let's prove that in QED we only need gauge-invariant counterterms by applying the BPHZ algorithm and an argument by induction.

1. Assume that we only need gauge-invariant CT up to $O(e^n)$. We want to show that this implies that gauge-invariant CT are sufficient to make the theory finite at $O(e^{n+1})$.
2. According to BPHZ, if we proceed to calculate diagrams up to $O(e^{n+1})$ we may encounter new divergences which can be absorbed by:
$$S[\Phi] \longrightarrow S[\Phi] + e^{n+1} S_{CT}^{(n+1)}[\Phi]$$
where $S_{CT}^{(n+1)}[\Phi]$ is a polynomial in Φ and $\partial_\mu \Phi$ of dim ≤ 4.
3. This will be a new contribution to Γ:
$$\Gamma[\bar{\Phi}] \longrightarrow \Gamma[\bar{\Phi}] + e^{n+1} S_{CT}^{(n+1)}[\bar{\Phi}]$$

4. The fact that $\Gamma[\bar{\Phi}]$ is made finite by adding $S_{CT}^{(n+1)}$ means that the divergent part of Γ has exactly that form.

$$\Gamma[\bar{\Phi}] = \Gamma_{finite}[\bar{\Phi}] - e^{n+1} S_{CT}^{(n+1)}[\bar{\Phi}]$$

5. We have proved that after a gauge transformation the change in $\Gamma[\bar{\Phi}]$ can only be of the form of the change of the GF-term. This can only be like

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}\mu^2(A_\mu A^\mu)$$

and doesn't have any power of e. Since we have assumed that $S_{CT}^{(n)}$ is gauge invariant, the WI are valid at this order and the only possible gauge-dependent part in Γ, having no additional power in e, cannot possibly affect $S_{CT}^{(n+1)}$.

6. We therefore conclude that $S_{CT}^{(n+1)}[\Phi]$ has to be gauge invariant.

Counterterms in QED (with $m_\gamma \neq 0$).

$$\mathcal{L}' = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}(1+A) + \bar{\psi}'(i\not{\partial} - e\not{A}')\psi'(1+B) - \bar{\psi}'\psi'(m+C)$$
$$+ \frac{1}{2}m_\gamma^2 A'_\mu A'^\mu - \frac{1}{2\xi}(\partial_\mu A'^\mu)^2$$

this is written in terms of renormalized fields ψ' and A'_μ related as usual to the bare fields:

$$\psi' = Z_2^{-1/2}\psi \quad \text{and} \quad A'_\mu = Z_3^{-1/2} A_\mu$$

which appear in the bare Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - e_0 A\!\!\!/ - m_0\right)\psi + \frac{1}{2} m_\gamma^{0\,2} A_\mu A^\mu - \frac{1}{2\xi_0}(\partial_\mu A^\mu)^2$$

By replacing the bare with the renormalized fields we get:

$$\mathcal{L} = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} Z_3 + Z_2\, \bar{\psi}'\left(i\partial\!\!\!/ - e_0 Z_3^{1/2} A\!\!\!/' - m_0\right)\psi' + \frac{1}{2} m_\gamma^{0\,2} Z_3 A'_\mu A'^\mu - \frac{1}{2\xi_0} Z_3 (\partial_\mu A'^\mu)^2$$

From which we can derive the following relations between renormalized and bare parameters:

$$1 + A = Z_3$$

$$1 + B = Z_2$$

$$e = Z_3^{1/2} e_0$$

$$m_0 = Z_2^{-1}(m + C)$$

$$m_\gamma = Z_3^{1/2} m_\gamma^0$$

$$\xi_0 = Z_3 \xi$$

Explicit examples of Ward identities.

Photon propagator in the presence of a mass:

$$\tilde{D}_{\mu\nu}(k) = -\frac{i}{k^2 - \mu^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] - \frac{i}{k^2/\xi - \mu^2}\left(\frac{k_\mu k_\nu}{k^2}\right) + O(e)$$

$\tilde{\Gamma}^{(0,0,2)}_{\mu\nu}$ is the inverse of $\tilde{D}_{\mu\nu}(k)$ (times i)

$$\tilde{\Gamma}^{(0,0,2)}_{\mu\nu} = -(k^2 - \mu^2)\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] - \left[\frac{k^2}{\xi} - \mu^2\right]\frac{k_\mu k_\nu}{k^2} + O(e)$$

$$\left[\text{Check: } -i\,\tilde{\Gamma}^{(0,0,2)}_{\mu\nu}\,\tilde{D}^{\nu\lambda} = \left(g_\mu{}^\lambda - \frac{k_\mu \cancel{k^\lambda}}{\cancel{k^2}}\right) + \frac{k_\mu \cancel{k^\lambda}}{\cancel{k^2}} + \mathcal{O}(e) = g_\mu{}^\lambda = \delta_\mu{}^\lambda \quad \checkmark\right]$$

$\Gamma^{(0,0,2)}_{\mu\nu}$ appears in

$$\Gamma = \frac{1}{2}\int d^4x_1\, d^4x_2\, A^\mu(x_1)\, A^\nu(x_2)\, \Gamma^{(0,0,2)}_{\mu\nu}(x_1,x_2) + \ldots$$

Gauge transf. of A_μ: $\delta A_\mu = \partial_\mu \delta\chi(x)$, indep. of e

$$\delta\Gamma = -\int d^4x_1\, d^4x_2 \left(\partial^\mu_1\, \Gamma^{(0,0,2)}_{\mu\nu}(x_1,x_2)\right) A^\nu(x_2)\, \delta\chi(x_1)$$

$$\Rightarrow \quad \partial^\mu_x\, \Gamma^{(0,0,2)}_{\mu\nu} = \partial^\mu_x\, \Gamma^{(0,0,2)}_{\mu\nu}\Big|_{e=0}$$

$$\text{or} \quad k^\mu\, \tilde{\Gamma}^{(0,0,2)}_{\mu\nu} = k^\mu\, \tilde{\Gamma}^{(0,0,2)}_{\mu\nu}\Big|_{e=0} = -\left(\frac{k^2}{\xi} - \mu^2\right) k_\nu$$

What is the meaning of this result if we look at $\Gamma^{(0,0,2)}_{\mu\nu}$ beyond $\mathcal{O}(e^0)$?

Consider the contribution of the self-energy, or vacuum polarization:

$$\equiv i\,\tilde{\Pi}_{\mu\nu}(k^2)$$

$$\tilde{D}_{\mu\nu} = \tilde{D}^T P^T_{\mu\nu} + \tilde{D}^L P^L_{\mu\nu}$$

$$\tilde{\Pi}_{\mu\nu} = \tilde{\Pi}^T P^T_{\mu\nu} + \tilde{\Pi}^L P^L_{\mu\nu}$$

} splitting in transverse and longitudinal components

$$P^T_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

$$\tilde{D}_{\mu\nu} = -\frac{i}{k^2 - \mu^2 - \tilde{\Pi}^T}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{i}{k^2/\xi - \mu^2 - \tilde{\Pi}^L}\left(\frac{k_\mu k_\nu}{k^2}\right)$$

$$\tilde{\Gamma}^{(0,0,2)}_{\mu\nu} = -\left(k^2 - \mu^2 - \tilde{\Pi}^T\right)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \left(\frac{k^2}{\xi} - \mu^2 - \tilde{\Pi}^L\right)\frac{k_\mu k_\nu}{k^2}$$

The Ward identity implies:

$$-\left(\frac{k^2}{\xi}-\mu^2-\tilde{\Pi}^L\right)k_\nu = -\left(\frac{k^2}{\xi}-\mu^2\right)k_\nu$$

$$\Longrightarrow \quad \tilde{\Pi}^L(k^2)=0$$

$$\Longrightarrow \quad \tilde{\Pi}_{\mu\nu} = \tilde{\Pi}^T P^T_{\mu\nu}$$

Ward identity for $\Gamma^{(1,1,1)}$.

$$\Gamma = \int d^4x\, d^4y\, \bar{\psi}(x)\psi(y)\,\Gamma^{(1,1,0)}(x,y) + \int d^4x\, d^4y\, d^4z\, \bar{\psi}(x)\psi(y)A^\mu(z)\,\Gamma^{(1,1,1)}(x,y,z) + \dots$$

Gauge transformation:

$$0 = \int d^4x\, d^4y\, \bar{\psi}(x)\psi(y)\left\{\Gamma^{(1,1,0)}(x,y)\left[ie\left(\delta\chi(x)-\delta\chi(y)\right)\right] + \int d^4z\left(\partial^\mu\delta\chi(z)\right)\Gamma^{(1,1,1)}(x,y,z)\right\} + \dots$$

All other terms have a different structure and cannot mix with these two, which by themselves must cancel each other. Integrating by parts the last term and considering the coefficient of $\delta\chi\cdot\bar{\psi}\psi$ we get

$$ie\,\Gamma^{(1,1,0)}(x,y)\left[\delta^4(x-z)-\delta^4(y-z)\right] = \partial^\mu_z\,\Gamma^{(1,1,1)}_\mu(x,y,z)$$

In momentum space this becomes

$$-ie\left[\tilde{S}_F^{-1}(\not{p}')-\tilde{S}^{-1}(\not{p})\right] = k^\mu\,\tilde{\Gamma}_\mu(p',p,k)$$

By differentiating this wrt k at $k_\mu=0$ at p fixed $\left(\frac{\partial}{\partial k^\mu}=\frac{\partial}{\partial p'^\mu}\right)$ we get:

$$-ie\frac{\partial}{\partial p'_\mu}\tilde{S}_F^{-1}(\not{p}')\Big|_{p'=p} = \frac{\partial}{\partial k_\mu}\left(k^\nu\,\tilde{\Gamma}_\nu(p',p,k)\right)\Big|_{k=0} \Longrightarrow -ie\frac{\partial}{\partial p_\mu}\tilde{S}_F^{-1}(\not{p}) = \tilde{\Gamma}_\mu(p,p,0)$$

$\Rightarrow$ in the limit $k_\mu \to 0$ inserting a photon line on an electron propagator is equivalent to differentiating the inverse of the electron propagator.

↳ this expression can be obtained by differentiating wrt $\frac{\delta}{\delta\bar{\psi}}$ and $\frac{\delta}{\delta\psi}$ the general WI

$$ie\left[\bar{\psi}(x)\frac{\delta\Gamma}{\delta\bar{\psi}(x)} - \frac{\delta\Gamma}{\delta\psi(x)}\psi(x)\right] - \partial^\mu\frac{\delta\Gamma}{\delta A_\mu(x)} = -\left[\frac{\Box}{\xi} + \mu^2\right]\partial_\mu A^\mu(x)$$