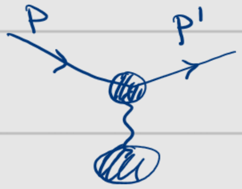
Amplitude for the emission of a soft photon in a generic process.

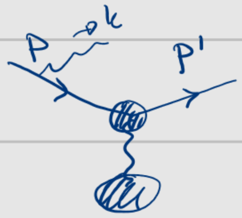
We discuss this phenomenon with the help of a concrete example:

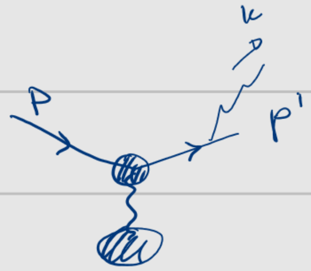
Scattering of an electron off an external field:



$$= \bar{u}(p')\, M_0(p',p)\, u(p)$$

where $M_0(p',p)$ is an amplitude which we don't need to specify for the moment. If we add a photon emission by the electron either before or after the interaction with the external field we get the following amplitude:





$$iM = -ie\,\bar{u}(p')\left[M_0(p',p-k)\,\frac{i(\not p - \not k + m)}{(p-k)^2-m^2}\,\not\varepsilon^*(k) + \not\varepsilon^*(k)\,\frac{i(\not p' + \not k + m)}{(p'+k)^2-m^2}\,M_0(p'+k,p)\right]u(p)$$

Let us consider the limit $|\vec{k}| \ll |\vec{p}-\vec{p}'|$ so that we can neglect k as four-momentum when compared to k or k' (or m).

$$M_0(p',p-k) \simeq M_0(p'+k,p) \simeq M_0(p',p)$$

and moreover $(\not p - \not k + m)\,\not\varepsilon^*(k)\,u(p) \simeq (\not p + m)\,\not\varepsilon^*(k)\,u(p) =$

$$= \left[2p\cdot\varepsilon^*(k) - \not\varepsilon^*(k)\,\cancel{(-\not p + m)}^{\,0}\right]u(p) = 2p\cdot\varepsilon^*(k)\,u(p)$$

as well as $\bar{u}(p')\,\not\varepsilon^*(k)\,(\not p' + \not k + m) = 2p'\cdot\varepsilon^*(k)\,\bar{u}(p')$.

For the denominators we write:

$$(p-k)^2 - m^2 = \cancel{m^2} - 2p\cdot k + k^2 \cancel{-m^2} \simeq -2p\cdot k$$

$$(p'+k)^2 - m^2 = 2p'\cdot k$$

and putting everything together we obtain:

$$i\mathcal{M} \simeq \bar{u}(p')\,\mathcal{M}_0(p',p)\,u(p)\cdot e\left(\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right) + \mathcal{O}(1)$$

So: the amplitude for the emission of a soft photon from an electron participating in a scattering process is given by the amplitude for the scattering process times a factor of $\mathcal{O}(\frac{1}{\omega})$, $(\omega = k^0)$ which is independent of the process, plus corrections of $\mathcal{O}(1)$.

If we take the modulus squared to obtain the cross section, we get the following:

$$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\sum_\lambda e^2\left|\frac{p'\cdot\varepsilon^{(\lambda)}}{p'\cdot k} - \frac{p\cdot\varepsilon^{(\lambda)}}{p\cdot k}\right|^2$$

$\Rightarrow$ Probability for emitting a soft photon:

$$d(\text{prob}) = \frac{d^3k}{(2\pi)^3}\sum_\lambda \frac{e^2}{2\omega}\left|\vec{\varepsilon}_\lambda\left(\frac{\vec{p}'}{p'\cdot k} - \frac{\vec{p}}{p\cdot k}\right)\right|^2$$

considering that the polarization vectors are transverse and have only spatial components

$$\text{Prob.} = \frac{\alpha}{4\pi^2}\int_0^{\omega_{max}} d\omega\cdot\omega^2\cdot\frac{1}{\omega^3}\underbrace{\sum_\lambda\int d\Omega_k\left|\vec{\varepsilon}_\lambda\left(\frac{\vec{p}'}{p'\cdot k/\omega} - \frac{\vec{p}}{p\cdot k/\omega}\right)\right|^2}_{4\pi I(\vec{v},\vec{v}')}$$

$4\pi I(\vec{v},\vec{v}') \leftarrow$ indep. of ω

$$= \frac{\alpha}{\pi} \int_0^{\omega_{max}} \frac{d\omega}{\omega}\, I(\vec{v},\vec{v}\,') \quad \leftarrow \text{ IR divergent } \quad (\omega \simeq 0)$$

The expression for the probability coincides with what one obtains in a classical calculation of the radiated energy which, divided by ω, gives the number of radiated γ.

We can make the integral finite by giving the photon a small mass m_γ:

$$\omega = \sqrt{m_\gamma^2 + \vec{k}^2} \quad \rightarrow \quad \int_0^{\omega_{max}} \frac{d\omega}{\omega} \rightarrow \int_{m_\gamma}^{\omega_{max}} \frac{d\omega}{\omega} = \ln\left(\frac{\omega_{max}}{m_\gamma}\right)$$

What is ω_{max}? $\quad \vec{p} = \vec{p}\,' + \vec{k} \Rightarrow |\vec{k}| = |\vec{p}-\vec{p}\,'| = |\vec{q}|$

If $|\vec{p}| = |\vec{p}\,'|$ (as in the $|\vec{k}| \ll |\vec{q}|$ approx), then

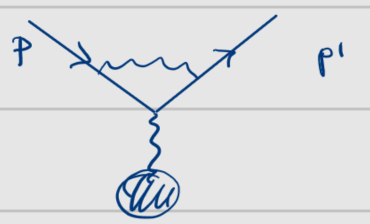
$$q^2 = (p_0 - p_0')^2 - \vec{q}^{\,2} = -\vec{q}^{\,2}$$

so that $\quad \ln\left(\frac{\omega_{max}}{m_\gamma}\right) = \frac{1}{2}\ln\left(\frac{-q^2}{m_\gamma^2}\right)$

$$\Rightarrow \text{Probability of soft photon emission} \simeq \frac{\alpha}{2\pi} \ln\left(\frac{-q^2}{m_\gamma^2}\right) I(\vec{v},\vec{v}\,')$$

$$\xrightarrow{-q^2 \to \infty} \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m_\gamma^2}\right) \ln\left(\frac{-q^2}{m^2}\right) \qquad \text{(Sudakov double logs)}$$

IR divergence in the vertex correction.



$$= \bar{u}(p')\, \Gamma_\mu\, u(p)$$

with $\Gamma_\mu = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$

(most general form of the vertex as implied by Lorentz and parity invariance as well as Ward ids.)

Outcome of the calculation: (Peskin-Schroeder 6.3)

$F_2(q^2)$ we have calculated. For $F_1(q^2)$ we write $F_1(q^2) = 1 + \delta F_1(q^2)$

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\Big((1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big)\right]$$

where Λ is a cut-off and $\Delta = -xyq^2 + (1-z)^2 m^2$.

By the renormalization condition $F_1(0) = 1$ we get rid of the cut-off Λ; we also add a photon mass m_γ to make the result IR-finite.

$$\Rightarrow \delta F_1^r(q^2) = \frac{\alpha}{2\pi}\int dx\,dy\,dz\,\delta(x+y+z-1)\left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \right.$$

$$\left. + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + m_\gamma^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + m_\gamma^2 z}\right]$$

What part of $\delta F_1^r(q^2)$ is IR divergent? It's easy to see that in the neighbourhood of $z=1$, so where $x=y=0$, the denominators of the second and third term in square brackets are proportional to m_γ^2. Let us look at these two terms only and evaluate the integral in dx:

$$\delta F_{1,IR}^r = \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\left[\frac{m^2(1-4z+z^2) + q^2(z+y)(1-y)}{m^2(1-z)^2 - q^2y(1-y-z) + m_\gamma^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + m_\gamma^2 z}\right]$$

Since we are interested in the IR-div, we can simplify the expression by setting $y=0$ and $z=1$ in the numerator. Moreover, m_γ is only there to remove the IR-div, and we can set its coefficient $z=1$.

$$\delta F^r_{1,IR} = \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy \left[\frac{-2m^2+q^2}{m^2(1-z)^2 - q^2 y(1-z-y) + m_\gamma^2} + \frac{2m^2}{m^2(1-z)^2 + m_\gamma^2} \right]$$

Variable change: $\quad y = (1-z)\xi\ , \quad w = 1-z$

$$\frac{d\xi}{dy} = \frac{1}{1-z}\ ; \quad \frac{d\xi}{dz} = \frac{y}{(1-z)^2} \qquad \det J = \frac{1}{z-1} = \frac{-1}{w}$$

$$\frac{dw}{dy} = 0 \qquad \frac{dw}{dz} = -1$$

$$dz\,dy = d\xi\, dw \cdot w = \frac{1}{2} d\xi \cdot dw^2$$

$$y(1-z-y) = (1-z)^2\left(\frac{y}{1-z} - \frac{y^2}{(1-z)^2}\right) = w^2\left(\xi - \xi^2\right)$$

$$\delta F^r_{1,IR} = \frac{\alpha}{4\pi}\int_0^1 d\xi \int_0^1 dw^2 \left[\frac{q^2-2m^2}{(m^2-q^2\xi(1-\xi))w^2 + m_\gamma^2} + \frac{2m^2}{m^2 w^2 + m_\gamma^2} \right]$$

$$= \frac{\alpha}{4\pi}\int_0^1 d\xi \left[\frac{q^2-2m^2}{m^2-q^2\xi(1-\xi)} \log\left(\frac{m^2-q^2\xi(1-\xi)}{m_\gamma^2}\right) + 2\log\left(\frac{m^2}{m_\gamma^2}\right) \right]$$

$$= -\frac{\alpha}{2\pi} \cdot f_{IR}(q^2) \cdot \log\left(\frac{-q^2}{m_\gamma^2}\right) + \text{"IR-finite"}$$

where
$$f_{IR}(q^2) = \int_0^1 d\xi \left[\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} \right] - 1$$

The IR-finite terms are of the form:

$$\frac{\alpha}{4\pi}\left\{ \int_0^1 d\xi \left[\frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)} \right] \log\left(\frac{m^2-q^2\xi(1-\xi)}{-q^2}\right) + 2\log\frac{m^2}{-q^2} \right\}$$

but won't be discussed any more. In a complete calculation that aims at a comparison with experiment all these need to be kept and evaluated in the final result.

If we evaluate the cross section we get:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega}\left[1-\frac{\alpha}{\pi}\left(f_{IR}(q^2)\ln\left(\frac{-q^2}{m_\gamma^2}\right) + \text{"IR-finite"}\right) + O(\alpha^2)\right]$$

In the limit $-q^2 \gg m^2$, we can easily evaluate the integral:

$$f_{IR}(q^2) = \frac{-q^2}{2}\int_0^1 d\xi \, \frac{1}{m^2-q^2\xi(1-\xi)}$$

$$\frac{1}{m^2-q^2\xi} + \frac{1}{m^2-q^2(1-\xi)} = \frac{m^2-q^2(1-\xi+\xi)}{(m^2-q^2\xi)(m^2-q^2(1-\xi))} =$$

$$= \frac{m^2-q^2}{m^4-m^2q^2(\cancel{\xi}+1-\cancel{\xi})+q^4\xi(1-\xi)} \simeq \frac{-q^2}{q^2\left(\frac{m^4}{q^2}-m^2+q^2\xi(1-\xi)\right)} =$$

$$= \frac{-1}{m^2-q^2\xi(1-\xi)}$$

$$\Rightarrow f_{IR}(q^2) = \frac{q^2}{2}\int_0^1 d\xi\left[\frac{1}{m^2-q^2\xi}+\frac{1}{m^2-q^2(1-\xi)}\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{-q^2}{m^2}\right)+\ln\left(\frac{-q^2}{m^2}\right)\right] = \ln\left(\frac{-q^2}{m^2}\right)$$

$$\Rightarrow F_1(q^2) \xrightarrow{-q^2\to\infty} 1-\frac{\alpha}{2\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{m_\gamma^2}\right)+O(\alpha^2)$$

$$\frac{d\sigma}{d\Omega}(p \to p') \simeq \frac{d\sigma_0}{d\Omega}\left[1 - \frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{m_\gamma^2}\right) + \mathcal{O}(\alpha^2)\right]$$

$$\frac{d\sigma}{d\Omega}(p \to p' + \gamma) \simeq \frac{d\sigma_0}{d\Omega}\left[+\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{m_\gamma^2}\right) + \mathcal{O}(\alpha^2)\right]$$

$$\frac{d\sigma}{d\Omega}(p \to p') + \frac{d\sigma}{d\Omega}(p \to p' + \gamma) = \frac{d\sigma_0}{d\Omega}\left(1 + \mathcal{O}(\alpha)\Big|_{\text{IR-finite}}\right)$$

Why should we consider the sum of these two cross sections which concern separate processes? The reason is that if the photon energy is too low my detector might be unable to detect it. Every detector has a certain sensitivity in energy, or a minimal energy of detection E_{min}, under which it is unable to detect photons. This means that what I can measure with my detector if I do not detect additional photons is the sum of two cross sections:

$$\frac{d\sigma}{d\Omega}(p \to p') + \int_0^{E_{min}} dE_\gamma \, \frac{d\sigma}{d\Omega}(p \to p' + \gamma) \equiv \left(\frac{d\sigma}{d\Omega}\right)_{meas}$$

Away from the limit $-q^2 \to \infty$ this looks as follows:

$$\left(\frac{d\sigma}{d\Omega}\right)_{meas} = \frac{d\sigma_0}{d\Omega}\left\{1 - \frac{\alpha}{\pi}\left[f_{IR}(q^2)\log\left(\frac{-q^2}{m_\gamma^2}\right) + \text{"IR-finite"}\right]\right.$$
$$\left. + \frac{\alpha}{2\pi}\left[I(\vec{v}, \vec{v}')\log\left(\frac{E_{min}^2}{m_\gamma^2}\right) + \text{"IR-finite"}\right] + \mathcal{O}(\alpha^2)\right\}$$

The expression for $I(\vec{v},\vec{v}')$ looked as follows:

$$I(\vec{v},\vec{v}') = \sum_\lambda \int \frac{d\Omega_k}{4\pi} \left| \varepsilon_\lambda \cdot \left(\frac{p'}{p'\cdot k/\omega} - \frac{p}{p\cdot k/\omega} \right) \right|^2 = -g_{\mu\nu} \int \frac{d\Omega_k}{4\pi} \left(\frac{p'}{p'\cdot \hat{k}} - \frac{p}{p\cdot \hat{k}} \right)^\mu \left(\frac{p'}{p'\cdot \hat{k}} - \frac{p}{p\cdot \hat{k}} \right)^\nu$$

$$= \int \frac{d\Omega_k}{4\pi} \left[\frac{2p\cdot p'}{(p'\cdot\hat{k})(p\cdot\hat{k})} - \frac{m^2}{(p'\cdot\hat{k})^2} - \frac{m^2}{(p\cdot\hat{k})^2} \right]$$

The second and third term are easy:

$$\frac{1}{4\pi}\int d\Omega_k \frac{m^2}{(\hat{p}\cdot k)^2} = \frac{m^2}{2}\int_{-1}^{1} d\cos\vartheta \frac{1}{(p_0 - |\vec{p}|\cos\vartheta)^2} = \frac{m^2}{\cancel{2}} \cdot \frac{\cancel{2}}{(p_0-|\vec{p}|)(p_0+|\vec{p}|)} = \frac{m^2}{p^2} = 1$$

For the first one we apply Feynman's trick to combine denominators:

$$\frac{1}{(p'\cdot\hat{k})(p\cdot\hat{k})} = \int_0^1 d\xi \frac{1}{(\xi p'\cdot\hat{k} + (1-\xi)p\cdot\hat{k})^2} = \int_0^1 d\xi \frac{1}{(p_\xi\cdot\hat{k})^2} \quad \text{with } p_\xi = \xi p' + (1-\xi)p$$

We then observe that the numerator $2p\cdot p' = -(p-p')^2 + 2m^2 = -q^2 + 2m^2$ is independent of $\cos\vartheta$ (and ϕ).

$$\Rightarrow \int \frac{d\Omega_k}{4\pi} \frac{2pp'}{(p'\hat{k})(p\hat{k})} = (-q^2+2m^2) \int_0^1 d\xi \int \frac{d\Omega_k}{4\pi} \frac{1}{(p_\xi\cdot\hat{k})^2} = \int_0^1 d\xi \frac{(-q^2+2m^2)}{p_\xi^2}$$

$$p_\xi^2 = (\xi p' + (1-\xi)p)^2 = \xi^2 m^2 + (1-\xi)^2 m^2 + 2\xi(1-\xi)p\cdot p'$$

$$= (1-2\xi+2\xi^2)m^2 + (\xi-\xi^2)(-q^2+2m^2)$$

$$= m^2 - \xi(1-\xi)q^2$$

All in all we obtain:

$$I(\vec{v},\vec{v}') = \int_0^1 d\xi \frac{-q^2+2m^2}{m^2-\xi(1-\xi)q^2} - 2 = 2 f_{IR}(q^2)$$

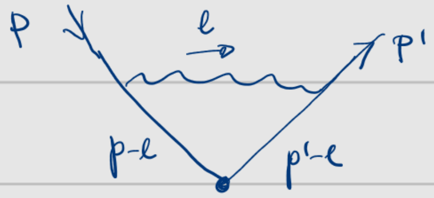
We conclude that

$$\left(\frac{d\sigma}{d\Omega}\right)_{meas} = \frac{d\sigma_0}{d\Omega}\left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln\left(\frac{-q^2}{E_{min}^2}\right) + \text{"IR-finite"} + O(\alpha^2)\right]$$

The log does not depend on m_γ any more, whereas "IR-finite" has a finite limit for $m_\gamma \to 0$ $\Rightarrow$ the observable is IR-finite.

This result is general and holds also beyond leading order, as discussed in Sect. 6.5 of Peskin-Schroeder.

The same principle holds also in non-abelian gauge theories like QCD.



$$\gamma^\alpha \int d^4\ell \quad \frac{1}{\ell^2} \cdot \frac{\not p' - \not\ell + m}{(p'-\ell)^2 - m^2} \, \gamma^\mu \, \frac{\not p - \not\ell + m}{(p-\ell)^2 - m^2} \, \gamma_\alpha \quad \sim \int \frac{d^4\ell}{\ell^4} \quad \text{for } \ell \sim 0$$

$$(p'-\ell)^2 - m^2 = -2p'\ell + \ell^2$$

$$(p-\ell)^2 - m^2 = -2p\cdot\ell + \ell^2$$