$$\vec{\mu}_e = g_e \frac{e}{2m_e} \vec{S}_e \qquad S_e = \frac{1}{2} \qquad g_e = 2$$ Uhlenbeck-Goudsmit (26)

Dirac theory explained $g_e = 2$

Twenty years later deviations from $g_e = 2$ were detected

$$a_e \equiv \frac{g_e - 2}{2} = 0.00118 \pm 0.00003$$ Kusch and Foley (47)

which can be understood in quantum electrodynamics (QED)

$$a_e = \frac{\alpha}{2\pi} = 0.00116$$ Schwinger (48)

and provided one of the first strong confirmations of QED

The latest measurement of a_e gives

Hanneke, Fogwell, Gabrielse (08)

$$a_e^{\mathrm{exp}} = 1\,159\,652\,180.73(28) \cdot 10^{-12}$$

In the standard model [Schwinger, Sommerfield, Petermann, Kinoshita et al., Remiddi et al.,...]

$$a_e^{\mathrm{SM}} = a_e^{\mathrm{QED}} + a_e^{\mathrm{had}} + a_e^{\mathrm{weak}} \qquad a_e^{\mathrm{QED}} = \frac{\alpha}{2\pi} + \sum_{n=2}^{\infty} a_e^{(2n)} \left(\frac{\alpha}{\pi}\right)^n$$

Kinoshita *et al.* (12) have recently evaluated $a_e^{(8,10)}$ leading to:

$$a_e^{\mathrm{SM}} = 1\,159\,652\,181.78(6)(4)(2)(77) \cdot 10^{-12}$$

with input: $\alpha^{-1} = 137.035\,999\,049(90)$

Bouchendira *et al.* PRL '11

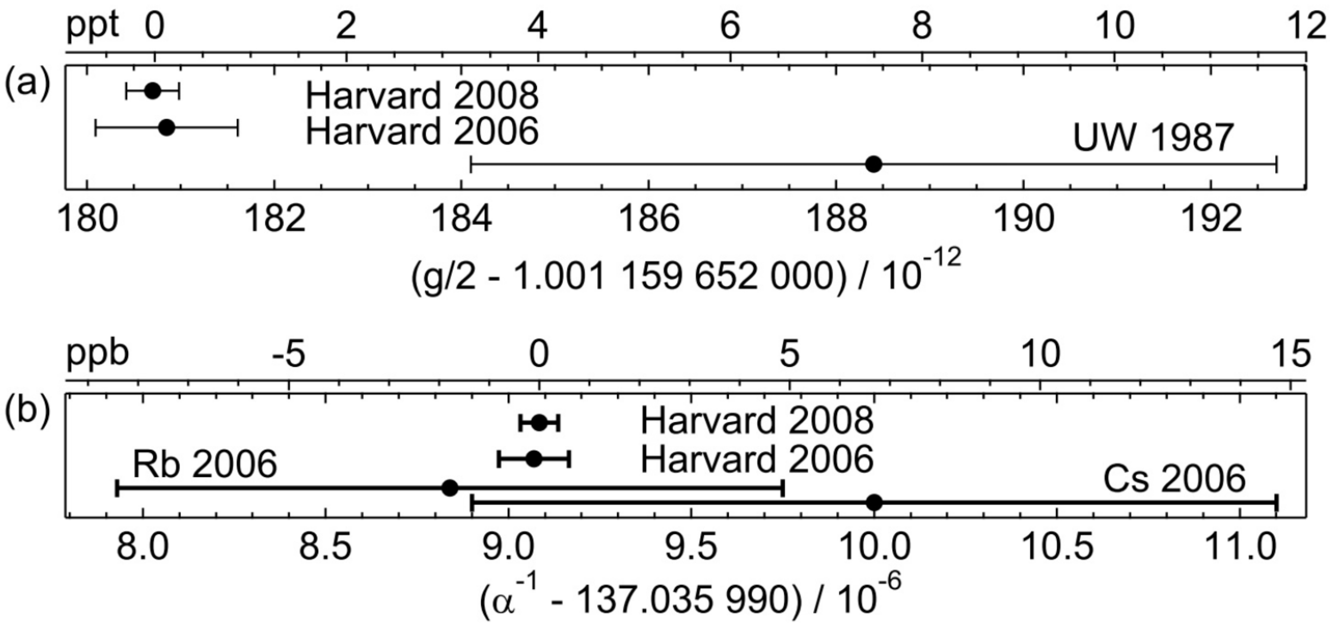
Hadronic and weak contributions have also been estimated

$$a_e^{\mathrm{hvp}} = 1.866(10)(5) \cdot 10^{-12} \qquad a_e^{\mathrm{hvp,NLO}} = -0.2234(12)(7) \cdot 10^{-12}$$

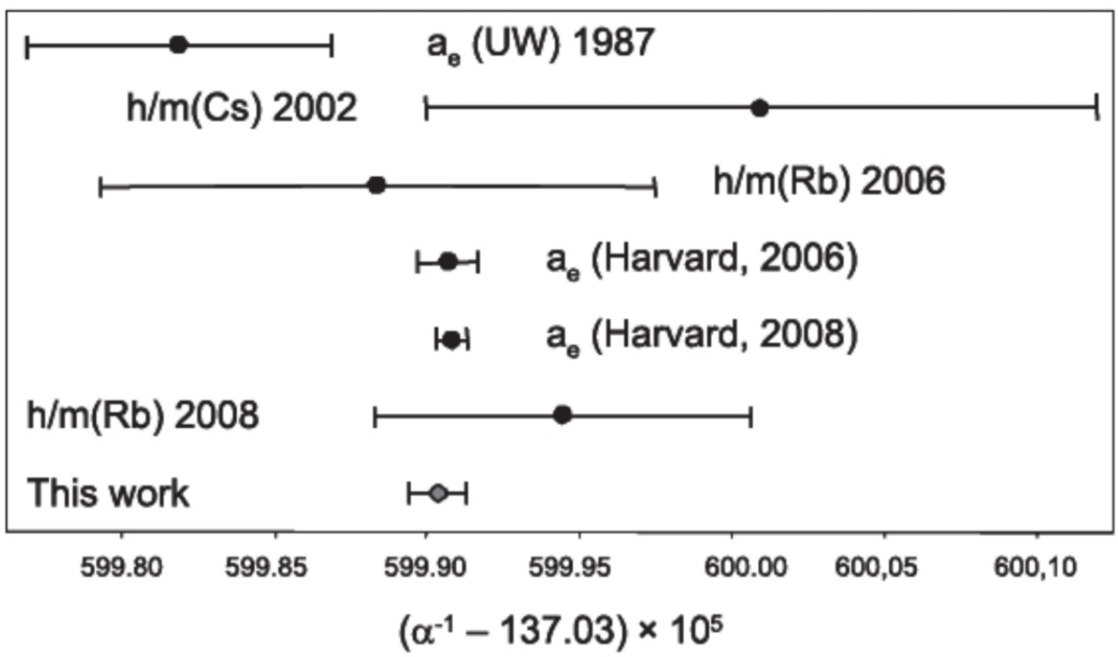
The latest measurement of a_e gives

Hanneke, Fogwell, Gabrielse (08)

$$a_e^{\rm exp} = 1\,159\,652\,180.73(28) \cdot 10^{-12}$$



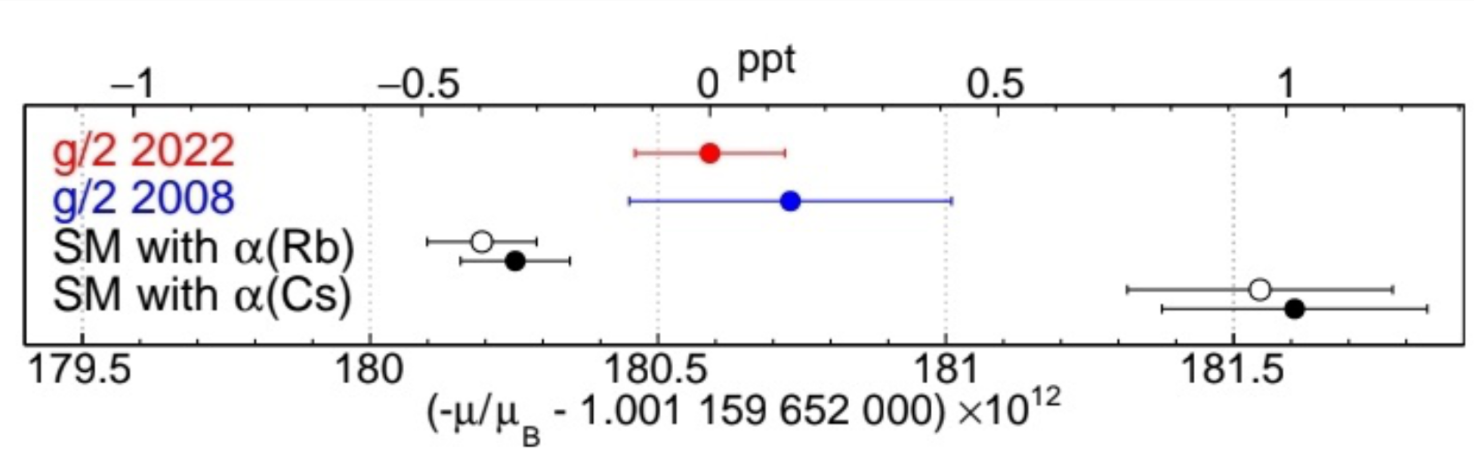
Recent improvements in alternative measurements of α

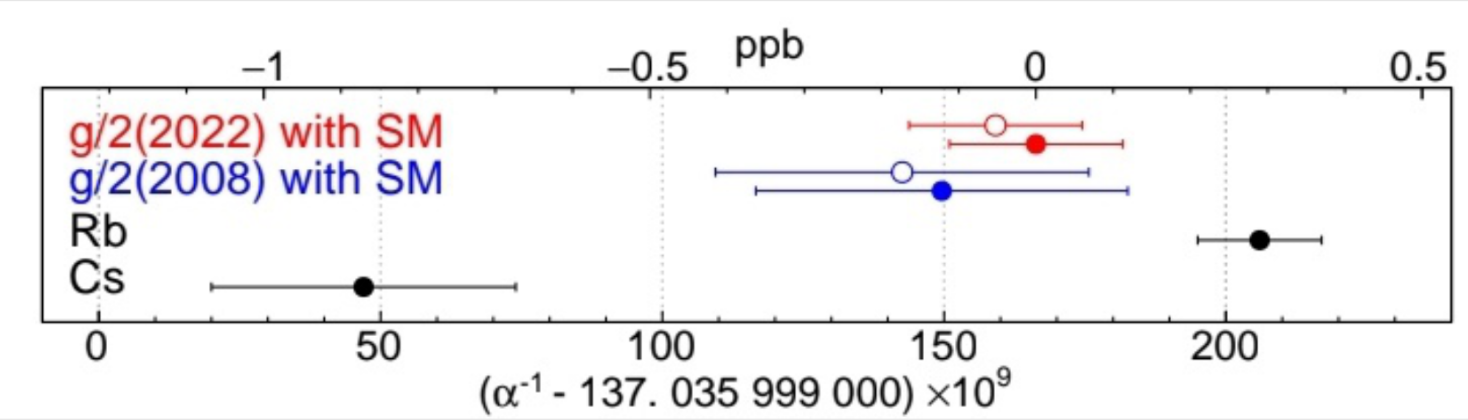


Bouchendira *et al.* PRL '11

PRL (2022) Fan, Myers, Sukra and Gabrielse

$$-\frac{\mu}{\mu_B} = \frac{g}{2} = 1.001\,159\,652\,180\,59\,(13) \quad [0.13 \text{ ppt}], \qquad (6)$$





The fine structure constant α is the fundamental measure of the strength of the electromagnetic interaction in the low energy limit. For the SI system of units, $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ is a measure of the vacuum permittivity ϵ_0, given that and e, $\hbar$ and the speed of light c are now defined [65]. Our μ/μ_B and the SM give

$$\begin{aligned} \alpha^{-1} &= 137.035\,999\,166\,(02)\,(15) \quad [0.014 \text{ ppb}]\,[0.11 \text{ ppb}], \\ &= 137.035\,999\,166\,(15) \qquad [0.11 \text{ ppb}], \qquad (8) \end{aligned}$$

with theoretical and experimental uncertainties in the first and second brackets. Figure 5 compares to the α measurements (black) that disagree with each other by

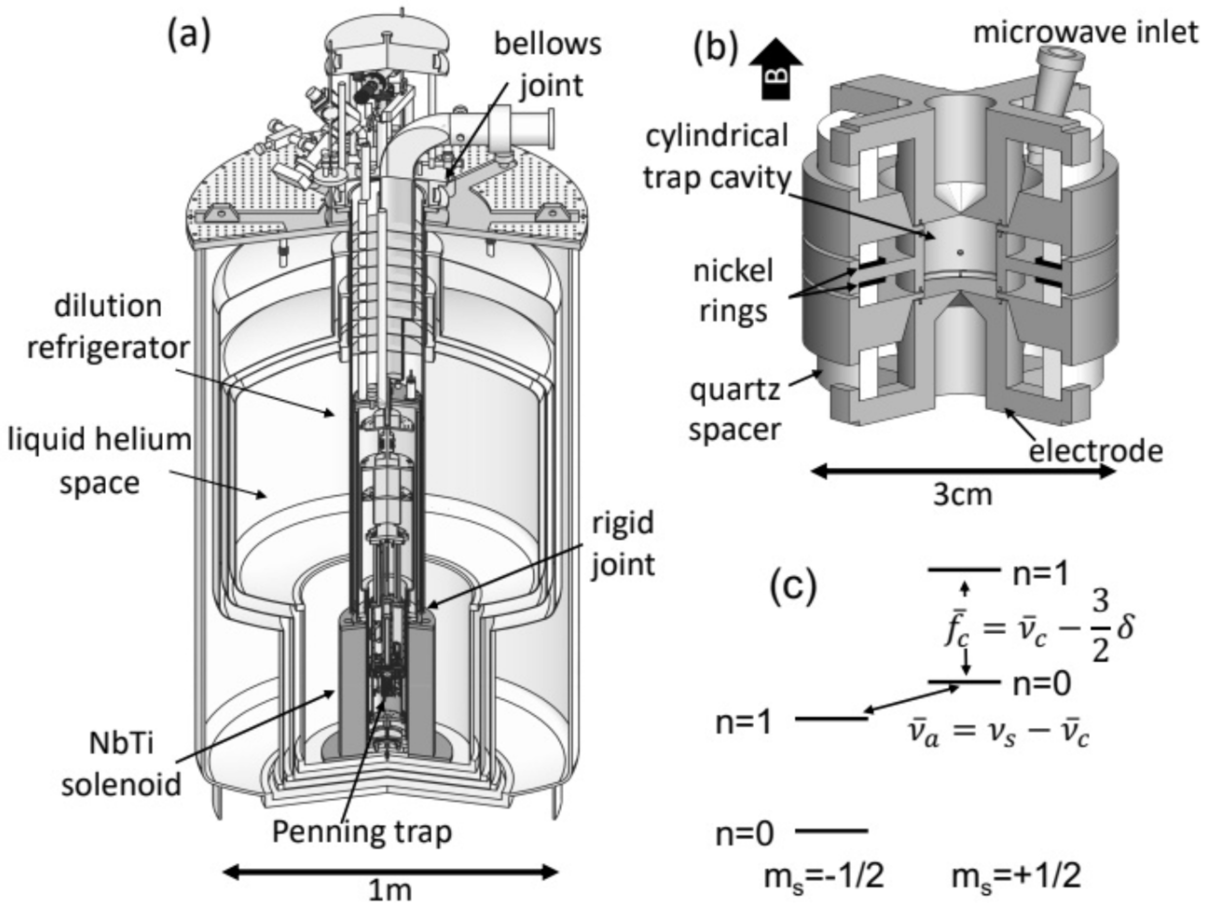
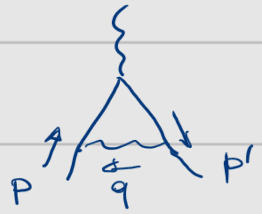


FIG. 2. (a) Cryogenic system supports a 50 mK electron trap upon a 4.2 K solenoid to provide a very stable B. (b) Silver electrodes of a cylindrical Penning trap. (c) Quantum spin and cyclotron energy levels used for measurement.

$$D = \underbrace{[(p'+q)^2 - m^2]}_{A}\underbrace{[(p+q)^2-m^2]}_{B}\underbrace{[q^2-M^2]}_{C}$$



$$\frac{1}{D} = \int_0^1 dx\,dy\,dz\;\delta(1-x-y-z)\;\frac{2}{[xA+yB+zC]^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \;\frac{2}{[xA+yB+(1-x-y)C]^3}$$

$$xA+yB+(1-x-y)C = x\,(p'^2+2p'q+q^2) + y\,(p^2+2pq+q^2) - m^2(y+x) + (1-x-y)q^2 - M^2(1-x-y)$$

$$= q^2 + 2(xp'+yp)\cdot q - M^2(1-x-y) \equiv \bar{D}$$

$$q' = q + xp' + yp$$

$$q'^2 = q^2 + (xp'+yp)^2 + 2(xp'+yp)\cdot q$$

$$= q^2 + (x^2+y^2)m^2 + 2xy\underbrace{pp'}_{m^2} + 2(xp'+yp)\cdot q$$

$$\bar{D} = q'^2 - m^2(x+y)^2 - M^2(1-x-y)$$

$$F_2(0) = 8ie^2m^2 \int_0^1 dx\int_0^{1-x} dy\;[(x+y)-(x+y)^2]\int \frac{d^4q}{(2\pi)^4}\frac{1}{[q^2-m^2(x+y)^2-M^2(1-x-y)]^3}$$

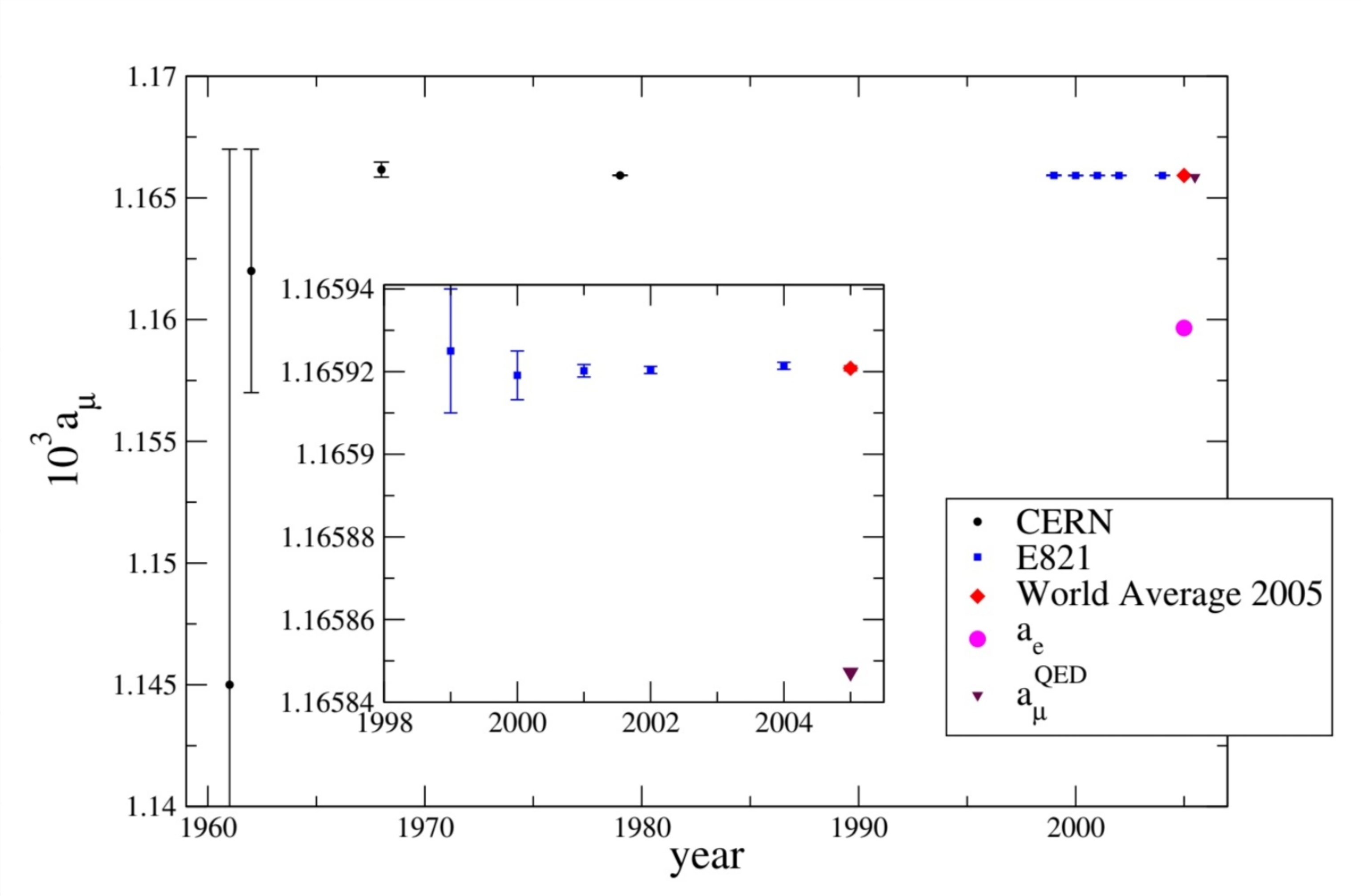
$$= \frac{8e^2m^2}{32\pi^2}\int_0^1 dx\int_0^{1-x} dy\,[(x+y)-(x+y)^2]\;\frac{1}{[m^2(x+y)^2+M^2(1-x-y)]}$$

$$= \frac{\alpha}{2\pi}\begin{cases} 1 & M=0 \\ \frac{2}{3}\frac{m^2}{M^2} & m \ll M \end{cases}$$

⇒ the contribution of a heavy particle to the anomalous magnetic moment of a lepton scales like:

$$\frac{a^H_{\ell_2}}{a^H_{\ell_1}} = \frac{m^2_{\ell_2}}{m_{\ell_1}}$$

for the muon is $4 \cdot 10^4$ larger than for the electron.



World Average (before FNAL)

$$a_\mu^{\rm exp} = (116\,592\,089 \pm 63) \times 10^{-11}$$

- The bulk of the difference between a_e and a_μ is due to QED and originates from large logs of m_μ/m_e

$$a_\mu^{\rm QED} - a_e^{\rm QED} = 619\,500.2 \times 10^{-11}$$
$$a_\mu^{\rm exp} - a_\mu^{\rm QED} = (7360 \pm 63) \times 10^{-11}$$

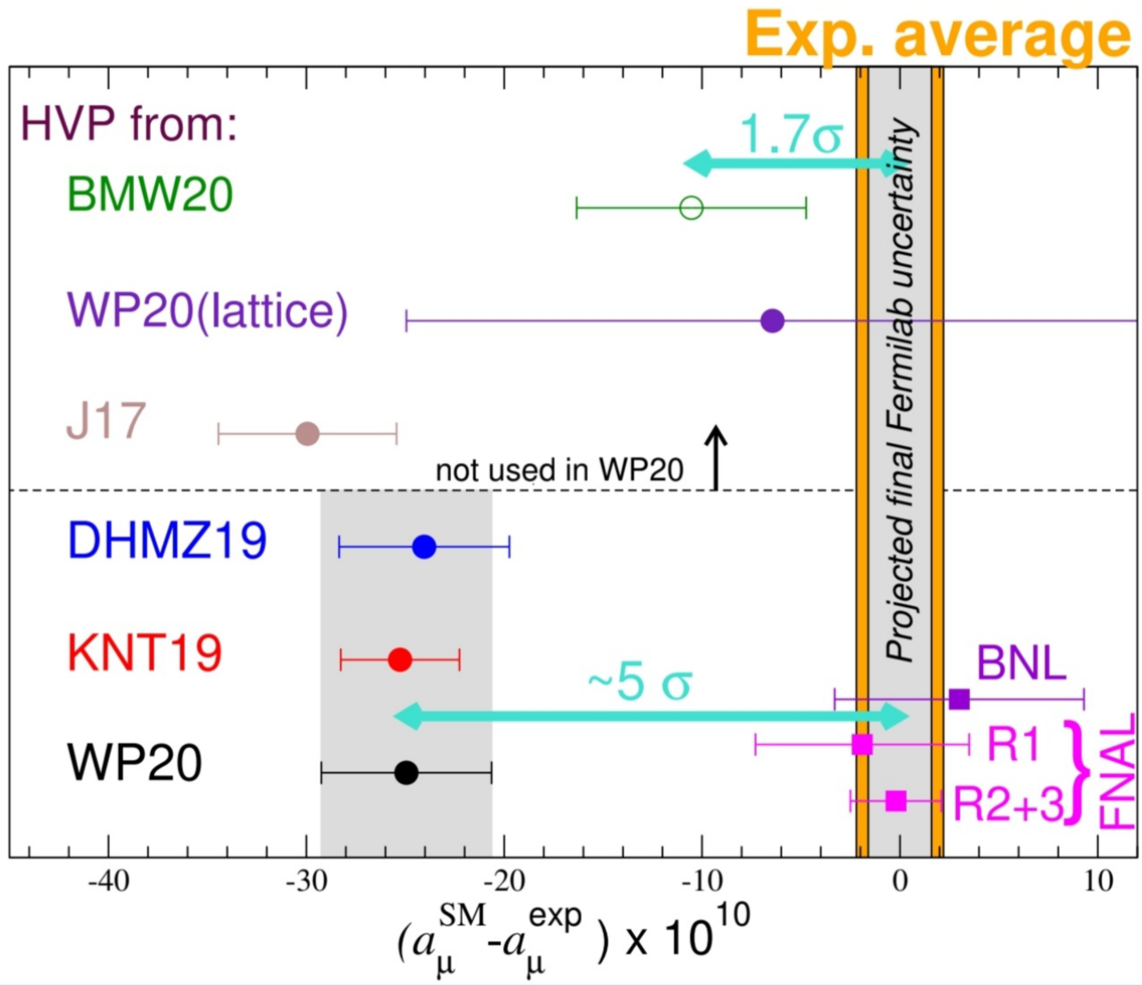
- Hadronic contributions are large

$$a_\mu^{\rm had} \simeq 7000 \times 10^{-11}$$

  "Seen" at the 5σ level already in 1979
- Weak contributions to a_μ

$$a_\mu^{\rm EW} = 154 \times 10^{-11} \simeq 2.5 \Delta a_\mu^{\rm exp}$$

After the 2023 Fermilab result



Calculation of the leading hadronic contribution (HVP).

where

$\mathcal{O}(\alpha_s^0)$ $\quad\quad$ $\mathcal{O}(\alpha_s)$ $\quad\quad$ $\mathcal{O}(\alpha_s^2)$ $\quad + \dots$

Expansion in α_s not meaningful because at low energy (here the relevant scale is $\sim m_\mu$) $\alpha_s \sim O(1)$.

How can this be calculated?

Let us look at the contribution of a lepton to vacuum polar.

Same calculation we did to obtain $a_e = \frac{\alpha}{2\pi}$, but now the photon propagator $\frac{1}{q^2}$ should be replaced by $\frac{1}{q^2} \cdot \Pi_e(q^2) \cdot \frac{1}{q^2}$.

Actually we need to enforce the condition $\Pi_e(q^2)=0$

so what we need to calculate is

$$\frac{1}{q^2}\,\overline{\Pi}_e(q^2)\,\frac{1}{q^2}$$

with $\quad \overline{\Pi}_e(q^2) = \Pi_e(q^2) - \Pi_e(0)$. $\overline{\Pi}_e(q^2)$ is finite.

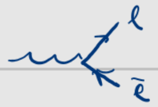
$$\tilde{\Pi}_e(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\left(\frac{m_e^2}{m_e^2 - x(1-x)q^2}\right)$$

Can we calculate the loop integral with this function inside the integrand? It looks difficult.

But we can take a different approach:

The imaginary part of $\tilde{\Pi}_e(q^2)$ is given by

$$\mathrm{Im}\,\tilde{\Pi}_e(q^2) = -\frac{\alpha}{3}\sqrt{1-\frac{4m_e^2}{q^2}}\left(1+\frac{2m_e^2}{q^2}\right)$$

which is related by the optical theorem to the modulus squared of the amplitude 

$$2\,\mathrm{Im}\Big[$$  $$\Big] =$$  $$= \int d\Pi\,\Big|$$  $$\Big|^2$$

From the imaginary part, which is related to the discontinuity across the cut, we can reconstruct the whole function, because this is analytic elsewhere:

$$\overline{\Pi}_e(q^2) = \frac{q^2}{\pi}\int_{4m_e^2}^{\infty} ds \, \frac{Im\,\overline{\Pi}_e(s)}{s(s-q^2)}$$

The dependence on q^2 of this function is now explicit and has the form of a propagator with mass s, $s \geq 4m_e^2$, over which we have to integrate at the end. But it is now clear that we can perform the loop integral because:

$$\frac{1}{q^2}\,\overline{\Pi}_e(q^2)\,\frac{1}{q^2} = -\frac{1}{\pi}\int_{4m_e^2}^{\infty} ds \, \frac{Im\,\overline{\Pi}_e(s)}{s} \cdot \frac{1}{q^2}\,\frac{1}{q^2-s}$$

so that

$$= \int ds \, \frac{Im\,\overline{\Pi}_e(s)}{s} \cdot \underbrace{\int \frac{d^4q}{(2\pi)^4}\,\frac{(\ldots\ldots)}{[1][2]\,q^2(q^2-s)}}_{\text{is finite and can be calculated analytically.}}$$

$$Im\,\overline{\Pi}_e(s) \propto \sigma(e^+e^- \to l^+l^-)$$

After performing the loop integral one gets an expression of the form:

$$a_\mu^{lvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_e^2}^{\infty} ds \, \frac{\hat{K}(s)}{s^2}\, R_e(s)$$

$$R_e(s) = \frac{\sigma(e^+e^- \to l^+l^-)}{4\pi\alpha^2/3s}$$

cross section in the limit $m_e \to 0$

To derive this formula we only have used unitarity (which relates the imaginary part of $\Pi(s)$ to a cross section) and analyticity (which allows us to reconstruct $\Pi(s)$ from its imaginary part). These properties are valid not only in QED, but in any QFT, even if perturbation theory cannot be applied. We can adopt it as it is and replace ℓ with any hadron which occurs in the spectrum of QCD:

$$a_\mu^{hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\hat{K}(s)}{s^2} R_h(s)$$

where $$R_h(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha^2/3s}$$