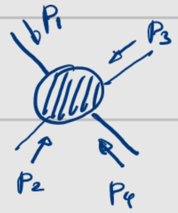
We want to study the behaviour of a QFT for large Euclidean momenta, or at short distances. In order to do this it is convenient, and in some cases even necessary (to avoid IR problems) to impose renormalization conditions which are independent of the masses of the theory one is considering. For large Euclidean momenta one expects the finite masses to play no role and we want to avoid that these could play an indirect role via the renorm. cond. We therefore adopt the following ones (considering a $\lambda\phi^4$ theory for simplicity):

$= 0 \qquad$ for $p^2 = -M^2$

$\dfrac{d}{dp^2}\Big($  $\Big) = 0 \qquad$ for $p^2 = -M^2$



$= -i\lambda \qquad$ for $(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -M^2$

M is the renormalization scale.

Notice that by the first two conditions we are imposing that the propagator of the renormalized field is that of a massless particle for $p^2 = -M^2$: $\qquad \langle 0|\,\phi(p)\,\phi(-p)\,|0\rangle = \dfrac{i}{p^2}$ at $p^2 = -M^2$.

The renormalized field is related to the bare one by

$$\phi = Z^{-1/2}\phi_0$$

$$\langle 0|\,\phi_0(p)\,\phi_0(-p)\,|0\rangle = \frac{iZ}{p^2} \qquad \text{at } p^2 = -M^2$$

Counterterm $\delta_Z = Z - 1$ as before, with the on-shell renorm.

(we have $\mathcal{L}_{CT} = \frac{1}{2}\delta_Z \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4$)

however, the expressions of $\delta_{Z,m,\lambda}$ are now different and depend on M.

This choice, however, is arbitrary and no physical result can depend on it. If we shift $M \to M'$, nothing should change. In particular our bare quantities, should know nothing about the renormalization scale.

Consider for example a (connected) Green's function:

$$G^{(n)}(x_1, \dots, x_n) = \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle_c = Z^{-n/2}\langle 0|T\phi_0(x_1)\dots\phi_0(x_n)|0\rangle_c$$

and a shift in the renormalization scale:

$$M \to M + \delta M$$

This will imply a corresponding shift in the coupling constant λ and in the renormalized field ϕ:

$$\lambda \to \lambda + \delta\lambda$$

$$\phi \to (1 + \delta\eta)\phi$$

The shift in the Green's function will be given fully by the shift in Z, since the bare Green's function cannot change as it doesn't know anything about M:

$$G^{(n)} \to (1 + n\,\delta\eta)\, G^{(n)}$$

At the same time $G^{(n)}$ is a function only of M and λ (other than the obvious spacetime arguments), so that its change has to be given by the change w.r.t. these two arguments:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M} \cdot \delta M + \frac{\partial G^{(n)}}{\partial \lambda} \cdot \delta \lambda = n \delta \eta G^{(n)}$$

Let us define:

$$\beta \equiv \frac{M}{\delta M} \cdot \delta \lambda \quad \text{and} \quad \gamma \equiv -\frac{M}{\delta M} \delta \eta$$

and multiply the eq. above by $\frac{M}{\delta M}$. We then obtain:

$$M \frac{\partial}{\partial M} G^{(n)} + \beta \frac{\partial G^{(n)}}{\partial \lambda} + n \gamma G^{(n)} = 0 \quad , \quad \text{with } G^{(n)}(x_1, \ldots, x_n; M, \lambda).$$

Clearly, the parameters β and γ do not depend on n and cannot depend on the spacetime arguments either. Since $G^{(n)}$ is renormalized, the cutoff has been removed already. Moreover we work in a massless theory so that the only scale is M => for dimensional reasons β and γ can only depend on the dimensionless coupling constant. More explicitly:

$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n \gamma(\lambda) \right] G^{(n)}(x_1, \ldots, x_n; M, \lambda) = 0$$

This is the Callan-Symanzik equation (1970).

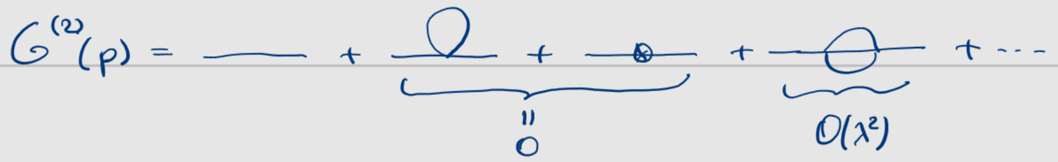
The generalization to QED is straightforward:

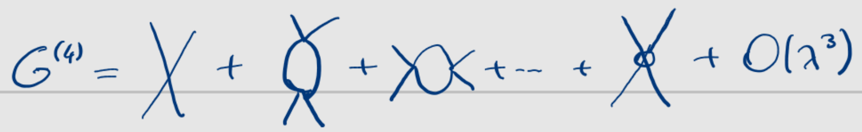
$$\left[M \frac{\partial}{\partial M} + \beta(e) \frac{\partial}{\partial e} + n \gamma_2(e) + m \gamma_3(e) \right] G^{(n,m)}(\{x_i\}; M, e) = 0$$

where n and m are the numbers of electron and photon fields in the Green's function, and γ_2 and γ_3 the corresponding rescaling functions.

Calculation of $\beta(\lambda)$ and $\gamma(\lambda)$ in $\lambda\phi^4$.

How does one proceed to calculate β and γ? One can calculate Green's functions in perturbation theory, and determine the expression of β and γ such that the Callan-Symanzik equation is satisfied to that order. Let us consider a concrete example:





$$G^{(4)} = \left\{ -i\lambda + (-i\lambda)^2 \left[iB_0(s) + iB_0(t) + iB_0(u) \right] - i\delta_\lambda \right\} \prod_{i=1}^{4} \frac{1}{p_i^2}$$

At $s = t = u = -M^2$ $\quad G^{(4)} = -i\lambda \prod_{i=1}^{4} \frac{1}{p_i^2}$

$$\Rightarrow (-i\lambda)^2 \left(3iB_0(-M^2) \right) - i\delta_\lambda = 0 \Rightarrow \delta_\lambda = -3\lambda^2 B_0(-M^2)$$

which gives

$$\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2} \left[\frac{1}{2 - d/2} - \log M^2 + \text{finite} \right]$$

The only M dependence is contained in this CT, and therefore we can calculate the derivative of $G^{(4)}$ wrt. M:

$$M\frac{\partial}{\partial M} G^{(4)} = \frac{\partial}{\partial \ln M} G^{(4)} = -i\,\frac{3\lambda^2}{(4\pi)^2}\cdot\prod_{i=1}^{4}\frac{1}{p_i^2}$$

By looking at $G^{(2)}$ it is clear that $\gamma = -M\frac{\delta\eta}{\delta M} = O(\lambda^2)$

Since $G^{(4)} = O(\lambda)$ the term proportional to γ is $O(\lambda^3)$ and the only one which can cancel $M\frac{\partial}{\partial M}G^{(4)}$ is the one proportional to β, which has to start at λ^2, and multiplies $\frac{\partial}{\partial\lambda}G^{(4)} = -i + O(\lambda)$.

The equation is satisfied with

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

Analogously, one can calculate the expression for γ at $O(\lambda^2)$, by considering the diagram  :

$$\gamma = \frac{\lambda^2}{12(4\pi)^4}$$

Let us consider the procedure of such a calculation for a more generic scalar theory:

$G^{(2)}(p) =$ $+$  $+$  $+ \dots$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A\log\frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}(i\delta_Z p^2)\frac{i}{p^2}$$

$$\delta_Z = A\log\frac{\Lambda^2}{M^2} + \text{finite}$$

and is the only dependence on M in $G^{(2)}(p)$

$$\Longrightarrow \quad -\frac{i}{p}M\frac{\partial}{\partial M}\delta_Z + 2\gamma\frac{i}{p^2} = 0$$

$$\Rightarrow \quad \gamma = \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z = -A$$

Notice that in solving this equation we have neglected the term proportional to β in the CS-eq., because this will be one order higher in the coupling constant.

Let us now consider β:

$$G^{(n)} = \left(\text{tree diagram}\right) + \left(\text{loop diagram}\right) + \left(\text{vertex CT}\right) + \left(\begin{matrix}\text{external leg}\\ \text{corrections}\end{matrix}\right)$$

$$= \prod_{i=1}^{n} \frac{i}{p_i^2} \left[-ig - iB \log \frac{\Lambda^2}{-p^2} - i\,\delta_g + (-ig) \sum_i \left(A_i \log \frac{\Lambda^2}{-p_i^2} - \delta_{Z_i} \right) \right] + \text{finite}$$

M-dependence

$$M \frac{\partial}{\partial M} \left(\delta_g - g \sum_i \delta_{Z_i} \right) + \beta(g) + g \sum_i \frac{1}{2} M \frac{\partial}{\partial M} \delta_{Z_i} = 0$$

$$\Rightarrow \quad \beta(g) = M \frac{\partial}{\partial M} \left(-\delta_g + \frac{1}{2} g \sum_i \delta_{Z_i} \right)$$

$$\delta_g = -B \log \frac{\Lambda^2}{M^2} + \text{finite}$$

$$\Rightarrow \quad \beta(g) = 2B - g \sum_i A_i$$

QED

An analogous calculation in QED gives

$$\gamma_2(e) = \frac{e^2}{16\pi^2} \quad ; \quad \gamma_3(e) = \frac{e^2}{12\pi^2}$$

$$\beta(e) = \frac{e^3}{12\pi^2}$$

A few remarks:

$$M\frac{\partial}{\partial M} = \frac{\partial}{\partial \ln M} \qquad \frac{\partial}{\partial M}\cdot\frac{\partial M}{\partial \ln M} = \frac{\partial}{\partial M}\cdot\frac{1}{\frac{\partial \ln M}{\partial M}} = \frac{\partial}{\partial M}\cdot\frac{1}{1/M} \checkmark$$

$$\beta(e) = M\frac{\partial e}{\partial M} = M\frac{\partial e}{\partial M}\cdot\frac{e}{e} = \frac{1}{2e}M\frac{\partial e^2}{\partial M} = \frac{2\pi}{e}M\frac{\partial \alpha}{\partial M}$$

$$\Rightarrow \beta(\alpha) = \frac{e}{2\pi}\beta(e) = \frac{e^4}{24\pi^3} = \frac{2\alpha^2}{3\pi} \qquad \Rightarrow \alpha(t) = \frac{\alpha(0)}{1-\frac{2\alpha(0)}{3\pi}t}$$

Solution of the Callan-Symanzik equation.

Consider $G^{(2)}(p)$ in a theory with a single scalar field:

$$G^{(2)}(p) = \frac{i}{p^2}\,g\left(-p^2/M^2\right)$$

because of dimensional analysis.

We can therefore use:

$$M\frac{\partial}{\partial M}G^{(2)}(p) = \frac{i}{p^2}M\frac{\partial}{\partial M}g\left(-p^2/M^2\right) = \frac{i}{p^2}\,g'(x)\Big|_{x=-p^2/M^2}\cdot\left(+\frac{2p^2}{M^2}\right)$$

having defined $p=\sqrt{-p^2}$

we also have $$p\frac{\partial}{\partial p}G^{(2)}(p) = \frac{i}{p^2}\left[-2g(x)\Big|_{x=-p^2/M^2} + g'(x)\Big|_{x=-p^2/M^2}\frac{2(-p^2)}{M^2}\right]$$

$$\Rightarrow M\frac{\partial}{\partial M}G^{(2)}(p) = \left(-p\frac{\partial}{\partial p} - 2\right)G^{(2)}(p)$$

So, the Callan-Symanzik eq. becomes:

$$\left[p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2 - 2\gamma(\lambda)\right]G^{(2)}(p) = 0$$

In a free theory $\beta(\lambda) = \gamma(\lambda) = 0 \Rightarrow G^{(2)}(p) = \frac{i}{p^2}$.

The full solution can be written as

$$G^{(2)}(p,\lambda) = \frac{i}{p^2}\, \mathcal{G}\left(\bar{\lambda}(p;\lambda)\right) \cdot \exp\left[2\int_0^t d\log(p'/M)\, \gamma(\bar{\lambda}(p';\lambda))\right]$$

where $\bar{\lambda}(p;\lambda)$ solves:

$$\frac{d}{d\log(p/M)} \bar{\lambda}(p;\lambda) = \beta(\bar{\lambda}) \qquad \text{where } \bar{\lambda}(M;\lambda) = \lambda$$

and $t = \log p/M$, and $\mathcal{G}(\lambda)$ is a function which must be determined by explicit calculation in perturbation theory.

In $\lambda\phi^4$ we have seen that $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$

$$\Rightarrow \frac{d}{d\log(p/M)} \bar{\lambda} = \frac{3\bar{\lambda}^2}{16\pi^2}$$

$$\left(\frac{3}{16\pi^2}\right)^{-1}\left[\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right] = \log\frac{p}{M}$$

$$\Rightarrow \boxed{\bar{\lambda}(p) = \frac{\lambda}{1 - \left(\frac{3\lambda}{16\pi^2}\right)\log p/M}}$$

Solution of the RGE.

28.5.24

Coleman's discussion of the solution of the RGE:

Consider the Fourier transform of a Green's function of a generic theory and factor out all "trivial" momentum dependence:

$$\text{F.T.} \langle 0|T\phi^{A_1}(x_1)\dots\phi^{A_s}(x_s)|0\rangle = \sum_r \underbrace{(\text{kinem. factors})^{(r)}}_{p^2, \not{p}, \dots} f^{(r)}(E/M, \Omega, g)$$

angular variables, kept fixed.

As we have seen in the example above the kinematic factors play no role in these equations, which can be written only for the scalar functions, and take the form:

$$\left[M\frac{\partial}{\partial M} + \beta^a(g)\frac{\partial}{\partial g^a} + \gamma\right] f(E/M, g) = 0$$

$$\beta^a(g) = M\cdot\frac{\partial g^a}{\partial M}$$

$$\gamma^A = \frac{1}{2}\frac{M}{Z^A}\frac{\partial Z^A}{\partial M}$$

We are now going to discuss the solutions assuming that $\vec{\beta}(\vec{g})$ and γ are known; $\gamma = \sum_A \gamma^A$ if there are more than one field.

Analogy to the evolution equation for the bacteria population in a fluid:

let $\rho(x,t)$ be the density

$v(x)$ the velocity with which the fluid flows down a tube

$L(x)$ the position-dependent illumination, which determines their rate of reproduction.

Equation for $\rho(x,t)$:

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right]\rho(x,t) = L(x)\,\rho(x,t)$$

$v(x) \longleftrightarrow \beta(x)$ in 1D, otherwise $\vec{v}(\vec{x}) \longleftrightarrow \vec{\beta}(\vec{x})$

$L(x) \longleftrightarrow -\gamma(x)$

Solution in two steps:

1. motion of an element of fluid

$$\frac{d\bar{x}(x,t)}{dt} = v(\vec{\bar{x}}) \qquad \text{with } \bar{x}(x,0) = x$$

$\bar{x}$ is a function of t for a given initial condition x, and it represents the position of an element of fluid which at time $t=0$ was placed at x. The solution tells us how an element of fluid moves if one knows the velocity field. The solution is not known in general, and can only be obtained if one knows the specific form of the velocity field $v(x)$.

2. evolution of the population inside an element of fluid.

$$\rho(x,0) = P(x) = P(\bar{x}(x,0))$$

At a given point the bacteria multiply exponentially depending on the value of $L(x)$, at the point where they are:

$$\left.\frac{d P(\bar{x}(x,t))}{dt}\right|_{t=0} = L(\bar{x}(x,t))\, P(\bar{x}(x,t))\Big|_{t=0}$$

is equivalent to the RGE above. The solution for all times is:

$$P(\bar{x}(x,t)) = P(\bar{x}(x,0)) \exp\left[\int_0^t dt'\, L(\bar{x}(x,t'))\right]$$

Translating this into an expression for $\rho(x,t)$, I would get:

$$\rho(\bar{x}(x,t),t) = \ldots \qquad \text{which is not what I want.}$$

But since $\bar{x}(x, t_1 - t_2)$ is equal to the position at time t_1 of the element of fluid which reaches x at time t_2, I can say that what I want is actually

$$\rho\left(\bar{x}(x, t_1 - t), t_1\right)\Big|_{t_1 = t}$$

The solution I want is obtained by inserting an offset of $-t$ in the argument of time of $\bar{x}$, since P is only a function of position:

$$\rho(x,t) = P(\bar{x}(x,-t)) \exp\left[\int_{-t}^{0} dt' L(\bar{x}(x,t'))\right]$$

If I set $t=0$ I obtain back the initial condition.

Having better understood the content of the differential equation, we can now apply it to our Green's functions. But before we do this we have to point out that the analogy is complete only if we set a correspondence between

$$t \to \ln(M/E)$$

The solution then reads as follows:

$$f(E/M, g) = F(\bar{g}(g,-t)) \exp\left[\int_{0}^{-t} dt' \gamma(\bar{g}(g,t'))\right]$$

where $\bar{g}(g,t)$ is the solution of the diff. eq.:

$$\frac{d\bar{g}}{dt} = \beta(\bar{g})$$

with boundary conditions $\bar{g}(g,0) = g$;

and where

$$F(\bar{g}(g,0)) = F(g) = f(1,g)$$

Let us now study the possible behaviours of β-functions.

We can calculate this function in perturbation theory only (on the blackboard or a piece of paper), which allows us to discuss the behaviour in the neighbourhood of $g=0$. There the β-function always has a zero, because in perturb. theory $\beta(g) = \mathcal{O}(g^2)$ or higher. If we move away from $g=0$ we have 3 possibilities:

1. $\beta(g) > 0$
2. $\beta(g) = 0$
3. $\beta(g) < 0$

1. If $\beta(g)>0$ it means that as we increase the scale M, g increases too: g is small in the IR and grows in the UV. As we consider processes of increasing energy the coupling constant becomes larger and larger, so that at some point the use of perturbation theory does not make sense any more. Examples:

$$\boxed{\bar{\lambda}(p) = \frac{\lambda}{1-\left(\frac{3\lambda}{16\pi^2}\right)\log p/M}} \quad \lambda\phi^4 \qquad\qquad \boxed{\alpha(t) = \frac{\alpha(0)}{1-\frac{2\alpha(0)}{3\pi}t}} \quad \text{QED}$$

2. If $\beta(g)=0$ in the neighbourhood of $g=0$ it means that we have no UV divergences in the perturbative expansion of the coupling constants. The latter does not flow, i.e. it stays constant, equal to its value in the IR. This is the case for so-called "finite" theories. Examples of such theories are known, but they are not relevant for phenomenology.

3. If $\beta(g)<0$ for $g \ll 1$ it means that as we increase M, the coupling constant decreases, and vanishes in the UV.

The running coupling constant in this case has a behaviour like:

$$\bar{g}^2(p) = \frac{g^2}{1+Cg^2\log(p/M)} \qquad \text{if } \beta(g) = -\frac{1}{2}Cg^3$$

If one reformulates the behaviour of the running coupling constant in terms of the bare coupling and a finite cutoff, a negative β-function corresponds to a situation in which the bare coupling goes to zero as we send the cutoff to infinity. At very short distances the theory behaves like a free theory and the interaction emerges only at finite distances.

This class of theories is called asymptotically free theories.

In $d=4$ only non-Abelian gauge theories are asympt. free.

Behaviour of the β-function for finite g.

If we increase g we can reach a point where perturbation theory does not work any more. The β-function is nonetheless well defined and can be calculated in principle with nonperturbative methods.

Even if we are not able to calculate the β-function in this region it is still worthwhile to analyze the possible different behaviours, at least qualitatively. We do this with the help of the figure below which is taken from Coleman.

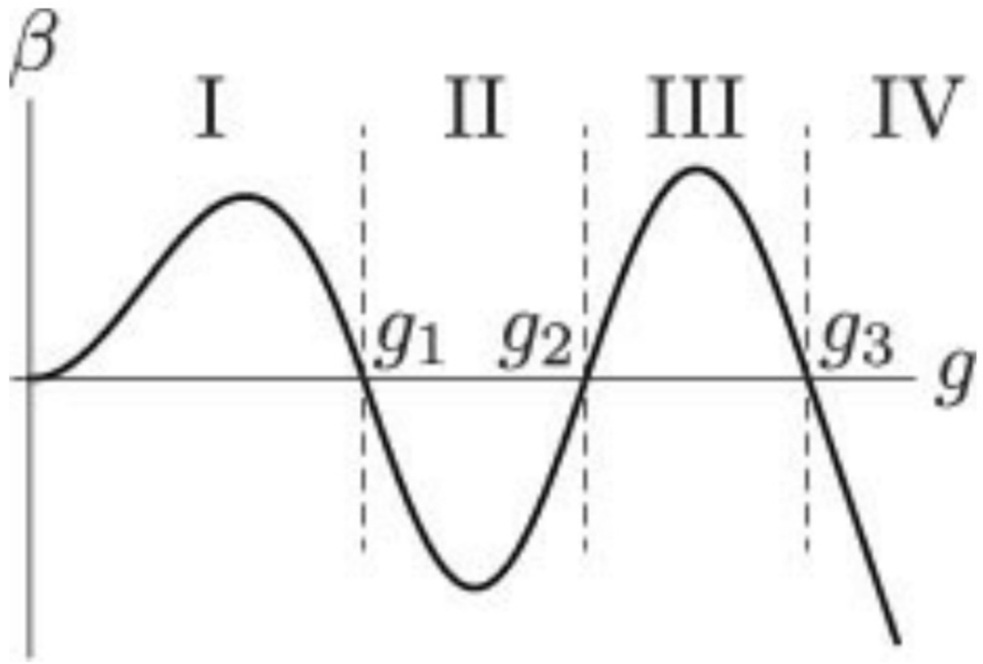


Figure 50.8: Hypothetical β function in φ^4 theory

[In the following we define $t = \ln E/M$]

In region I $\beta(g) > 0 \Rightarrow \bar{g}(g,t)$ increases always, but reaches a maximum value $\bar{g}(g,t) \xrightarrow{t\to\infty} g_1$.

In the plot above, the zero at g_1 is reached linearly, so that the behaviour in the neighbourhood of g_1 can be described as:

$$\beta(\bar{g}) = -a(\bar{g} - g_1) + O((\bar{g} - g_1)^2)$$

In this region the solution of the RGE reads

$$\frac{d\bar{g}}{dt} = -a(\bar{g} - g_1) \qquad (\bar{g} - g_1) = C \cdot e^{-at}$$

$$\Rightarrow \quad \bar{g}(t) = g_1 + O(e^{-at})$$

It reaches the value of g_1 exponentially fast in t.

In this region we can also analyze the behaviour of the Green's functions.

$$f(E/M,g) = F(\bar{g}, \ln(E/M)) \exp\left[\int_0^{\ln(E/M)} dt' \, \gamma(\bar{g}(g,t'))\right]$$

$$f(E/M,g) \longrightarrow F(g_1) \cdot \exp[\cdots]$$

For the argument of the exp. we can write:

$$\int_0^{\ln(E/M)} dt' \, \gamma(\bar{g}(g,t')) = \int_0^{\infty} dt' \left[\gamma(\bar{g}(g,t')) - \gamma(g_1)\right] + \int_0^{\ln E/M} dt' \, \gamma(g_1)$$

$$= \ln(k) + \ln E/M \cdot \gamma(g_1)$$

$$\Longrightarrow \quad f(E/M,g) = F(g_1)\, k \, (E/M)^{\gamma(g_1)}$$

Remember that the any obvious dependence on the momenta has been already pulled out of the Green's function, so that this additional "power dependence" on the moment is not what one would expect from a pure dimensional analysis. The fields scale with energy differently than how one would expect from their energy dimensions, an effect which is due to the interaction. The functions γ are called "anomalous dimensions" for this reason.

In region II $\beta(g) < 0$, which means that as we increase t the coupling constant moves to the left. Starting from any value between g_1 and g_2 the evolution of $\bar{g}(t)$ with t brings us close to g_1. In particular, no matter how close to g_2 we start, if are to its left we move towards g_1.

Of course, if we are at g_2, $\beta(g_2)=0$ and we stay there.
g_1 is called an UV stable fixed point, whereas g_2 is an unstable one.

If we move to region III we see that the situation is analogous to region I: the evolution of the coupling constant brings it away from g_2 (where it would go in the IR) and towards g_3, which is another UV stable fixed point.

Are all behaviours of the coupling constant equally acceptable?
No, that the coupling constant becomes infinite in the UV, or for short distances is physically unacceptable, whereas the opposite behaviour, i.e. vanishing coupling constant at short distance and a buildup of the charge due to the interaction makes physical sense. The only theories which have a negative β-function (in D=4) are non-Abelian gauge theories and the hope is that nature can be described with these. Indeed the SM is a combination of gauge theories, unfortunately, one of them is Abelian and has a positive β-function. Note, however, that

$$\alpha(0) = \frac{1}{137} \quad \text{whereas} \quad \alpha(M_Z) \simeq \frac{1}{128}$$

so that a problem with a Landau pole only shows up at unreachable energies. And to solve the problem completely, the idea of a so-called GUT or Grand-Unified Theory was formulated. In this scenario

QED emerges out of the breaking of a larger, non-Abelian gauge group. Above the energy scale where the breaking happens, one would have to deal with an unbroken non-Abelian gauge theory only, and therefore with an asymptotically free one.

Unfortunately the three coupling constants of the SM do not meet at the same point. For this one needs SUSY. For many years this was considered as a strong hint that SUSY particles had to be found at the LHC. The search was not successful so far, but it continues...